CHAPTER

5

Systems of Equations and Inequalities

Overview

In Chapter 5, students look at systems of linear equations and consider linear inequalities. Then they put these two ideas together to think about systems of linear inequalities. In **Lessons 5.1** through **5.4,** students learn five ways to solve a system of equations: tables, graphs, the substitution method, the elimination method, and row operations on matrices. Inequalities in one variable are introduced in **Lesson 5.5;** students perform operations on inequalities and learn why multiplying an inequality by a negative number reverses the direction of the inequality. In **Lesson 5.6,** students learn how to graph inequalities in two variables and how to check whether given points are solutions. In **Lesson 5.7,** students graph and solve systems of inequalities.

The Mathematics

Systems of Linear Equations

A linear equation in two variables has infinitely many solutions. Each solution is an ordered pair of numbers. If you substitute those numbers for the two variables, the equation becomes a true statement. A *solution* to a **system** of two such equations is an ordered pair that is a solution to each equation.

A system may have zero solutions, one solution, or infinitely many solutions. In the first case, the equations are called *inconsistent*. In the last, they're called *redundant.*

A system of two linear equations in two variables can be solved using many methods.

Graphing. If you graph the line represented by each equation, the point of intersection will have coordinates that give a solution to (that is, *satisfy*) both equations. If the lines are parallel, there is no solution. If the lines coincide, the system has infinitely many solutions.

Tables. To find a solution, if there is one, look for a pair of numbers that appears in the table of solution values for each equation.

Method of substitution. For an exact solution to a system, solve one equation for one of the variables in terms of the other variable, substitute the resulting expression for that variable in the other equation, solve that linear equation in the one remaining variable, substitute the solution into either original equation, and solve for the other variable.

Method of elimination. If the coefficients make it difficult to solve either equation for a variable, multiply one or both equations by a constant to get the opposite coefficients for the same variable in the two equations. Then add the equations to eliminate that variable, leaving a single linear equation in the other variable. Substitute the solution to that equation into either original equation, which can then be solved for the other variable.

Row reduction of a matrix. Put the coefficients of the two equations into a matrix and then perform row operations that mimic the steps in the method of elimination. A procedure that systematically obtains a diagonal matrix from which the solution can be easily read is called *Gaussian elimination.* Calculators and computers solve systems of equations by using variations on this method.

Linear Inequalities

An inequality is like an equation except that the equal sign is replaced by $<$ (less than), $>$ (greater than), $\leq$ (less than or equal to), or $\geq$ (greater than or equal to). Statements using $<$ or $>$ are called *strict inequalities.*

You can solve linear inequalities in one variable as you would solve linear equations by balancing, with the exception that if you multiply or divide by a negative number, you must switch the direction of the inequality. The solution can be graphed as

a ray on a number line. The endpoint of the ray is either an open circle (for strict inequalities) or a solid circle.

Solutions to an inequality in two variables can be graphed on a plane. First you write an equation by replacing the inequality symbol with an equal sign. Graph that equation. If the inequality is strict, draw a dashed line; otherwise, draw a solid line. The solutions to the inequality form a half-plane on one side of that line. Shade in the solution region.

The solutions to a system of inequalities in two variables are graphed as the intersection of the half-planes representing the solutions of the individual inequalities in the system. Such a graph provides a way of solving simple problems in the area of mathematics called *linear programming.* In that field, the inequalities are called *constraints,* and the solution set is called a *feasible region.* An example of a linear programming problem appears in Take Another Look on page 330. Linear programming is discussed more extensively in *Discovering Advanced Algebra*.

Using This Chapter

If you prefer not to introduce all five methods for solving systems of equations, the most important methods are in the first two lessons. Lesson 5.4 assumes students have used matrices before.

Resources

Discovering Algebra Resources

Teaching and Worksheet Masters
Lessons 5.2, 5.4–5.7

Calculator Notes 1B, 3B, 4B, 5A, 5B, 5C, 7D

Sketchpad Demonstrations
Lessons 5.1, 5.6, 5.7

Fathom Demonstration
Lesson 5.2

CBL 2 Demonstration
Lesson 5.1

Dynamic Algebra Explorations online
Lessons 5.3, 5.5, 5.8

Assessment Resources
Quiz 1 (Lessons 5.1–5.4)
Quiz 2 (Lessons 5.5–5.7)
Chapter 5 Test
Chapter 5 Constructive Assessment Options

More Practice Your Skills for Chapter 5

Condensed Lessons for Chapter 5

Other Resources

Chinese Mathematics: A Concise History by Lǐ Yǎn and Dù Shírán.

Jinkōki by the Waasen Institute.

For complete references to these and other resources, see www.keypress.com/DA.

Materials

- watches with second hands
- masking tape, 6 m ropes, or chalk
- motion sensors, *optional*
- ropes of different thickness, each 1 m long (two per group)
- meterstick or tape measure
- 9 m thin rope, *optional*
- 10 m thick rope, *optional*
- 30 paper clips, all the same size
- pennies (120)
- 8.5-by-11 in. sheets of paper
- 20–40 ft rope marked with whole numbers from −10 to 10, or paper or chalk number line with two markers, *optional*

Pacing Guide

	day 1	day 2	day 3	day 4	day 5	day 6	day 7	day 8	day 9	day 10
standard	5.1	5.2	5.2	5.3	5.3	5.4	5.4	quiz, 5.5	5.6	5.7
enriched	5.1	5.2	5.2	5.3	5.3	5.4	5.4	quiz, 5.5	5.6, project	5.7
block	5.1	5.2	5.3	5.4, quiz	5.5, 5.6	5.7, review	assessment			

	day 11	day 12	day 13	day 14	day 15	day 16	day 17	day 18	day 19	day 20
standard	quiz, review	assessment								
enriched	review, TAL	assessment								

CHAPTER 5 OBJECTIVES

- Model real-world situations with systems of two linear equations in two variables
- Approximate solutions to systems of two linear equations using tables and graphs, understanding how the relative position of the lines indicates the number of solutions to the system
- Solve systems of linear equations using substitution, elimination, and row operations on a matrix (Gaussian elimination)
- Model real-world situations with one-variable inequalities
- Solve one-variable inequalities, including use of the sign-change rule when multiplying or dividing both sides by a negative number
- Graph solutions to one-variable inequalities on a number line, showing whether they are strict inequalities
- Model real-world problems with two-variable inequalities and show their solutions as half-planes on the coordinate plane
- Model real-world problems with systems of two-variable inequalities and show the solutions as the intersection of two or more half-planes

CHAPTER

5

Systems of Equations and Inequalities

Freshly painted umbrellas dry in the sun outside the Nagatsu factory in Kyushu, Japan. The sticks in their frames form intersecting lines like the graphs of linear equations. Where do you see only two lines intersecting at a point? Where do several lines intersect?

OBJECTIVES

In this chapter you will

- learn to solve systems of linear equations
- solve systems using the substitution method
- solve systems using the elimination method
- solve systems using matrices
- graph inequalities in one and two variables
- solve systems of linear inequalities

This photo suggests many lines graphed on a plane, and the points of their intersection suggest solutions to systems of equations. As you look at this image with your students, you can look at it as a two-dimensional image or imagine the three-dimensional space in which the umbrellas exist. Looked at as a two-dimensional picture, the lines formed by the mechanism that opens and supports the umbrella cross the lines of the umbrella structure. What other lines intersect? What lines intersect in the three-dimensional space? At which points do several lines intersect?

LESSON

5.1

Solving Systems of Equations

In previous chapters you studied linear relationships in the contexts of elevators, wind chill, rope length, and walks. In this chapter you'll consider two or more linear equations together. A **system of equations** is a set of two or more equations that have variables in common. The common variables relate similar quantities. You can think of an equation as a condition imposed on one or more variables, and a system as several conditions imposed simultaneously.

When solving a system of equations, you look for a solution that makes each equation true. There are several strategies you can use. In this lesson you will solve systems using tables and graphs.

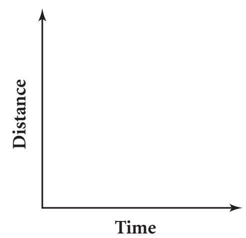
Investigation
Where Will They Meet?

You will need

- one motion sensor
- a tape measure or chalk to make a 6-meter line segment

In this investigation you'll solve a system of simultaneous equations to find the time and distance at which two walkers meet.

Suppose that two people begin walking in the same direction at different average speeds. The faster walker starts behind the slower walker. When and where will the faster walker overtake the slower walker?



Step 1 Sketch a graph showing both walks. Which line represents the faster walker?

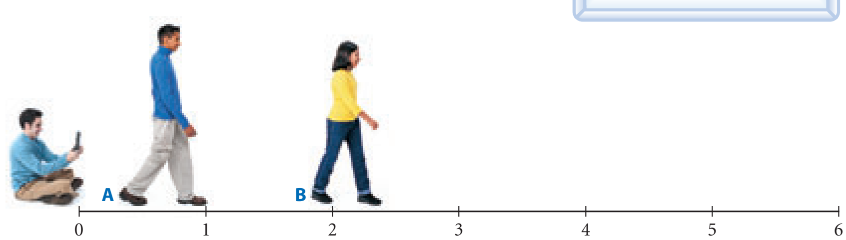
Now act out the walk.

Step 2 Mark a 6 m segment at 1 m intervals. In your group, designate Walkers A and B, a timekeeper, and a recorder.

Step 3 Practice these walks: Walker A starts at the 0.5 m mark and walks toward the 6 m mark at a speed of 1 m/s. Walker B starts at the 2 m mark and walks toward the 6 m mark at 0.5 m/s.

Procedure Note

The timekeeper counts each second out loud. The walkers walk at the given speeds by noting their positions on the marked segment. The recorder uses a motion sensor to measure the time and position of each walker.



LESSON

5.1

PLANNING

LESSON OUTLINE

One day:

- 20 min Investigation
- 10 min Example
- 10 min Sharing
- 5 min Closing
- 5 min Exercises

MATERIALS

- watch with second hand (one per group)
- 6 m or 6 yd path with 1 m or 1 yd marks on the floor or ground (masking tape or 6 m ropes with 1 m marks are good inside; use chalk or the yard markers on a football field outside)
- motion sensors (one per group), *optional*
- Calculator Notes 1B, 3B, 7D
- Sketchpad demonstration Solving Systems of Equations, *optional*
- CBL 2 demonstration Hot and Cold, *optional*

TEACHING

A system of two linear equations can model some real-world problems. These systems can be solved using graphs or tables.

Guiding the Investigation

In the step-by-step investigation, both walkers travel in the same direction, so their slopes have the same sign. The example describes a situation in which two people walk in opposite directions, so the two slopes have opposite signs.

See page 274 for the answer to Step 1.

NCTM STANDARDS

CONTENT		PROCESS	
	Number		Problem Solving
✓	Algebra	✓	Reasoning
	Geometry		Communication
✓	Measurement	✓	Connections
	Data/Probability	✓	Representation

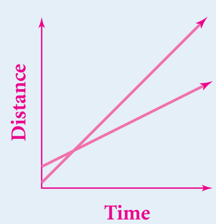
LESSON OBJECTIVES

- Model real-world situations with systems of two equations
- Solve systems of two linear equations using tables and graphs
- Understand that the intersection of the two lines provides a solution to the system and thus to the real-world problem

One Step
Ask that one student in each group walk toward a point (the motion sensor, if you are using one) from 6 m away at a rate of 1 m/s, while another student walks away from that point, beginning 2 m away, at 0.5 m/s. The group is to represent the situation, and the meeting point, in as many ways as it can. Encourage students to use graphs and pairs of equations and to think about rates at which the two walkers are closing the distance between them. As groups accomplish this task, ask them to repeat the process, but this time with the first person beginning 0.5 m away from the given point and walking *away* from it at 1 m/s.

Step 1 As needed, remind students that a graph sketch needs to have labels on the axes. Students may need clarification that the time is the number of seconds elapsed since the walkers started and the distance is the number of meters the walkers are from one end of the path.

Step 1



The steeper line represents the faster walker.

Step 2 In a group of three, one person will need to be both a walker and the timekeeper.

Steps 3 and 4 Before students start recording data, they should practice the walks until they know how fast to move. Even with experience, they might want to do each walk three times and average the results.

The recorder should stand at the 0 m mark and hold the sensor to collect the data on each walker separately. See Calculator Note 3B to learn how to collect the data.

Step 4 When the walkers can follow the walk instructions accurately, record and download the motion of each walker as a separate event. First record Walker A's motion with the motion sensor. Download Walker A's data to a graphing calculator and move it to other lists. [▶ See **Calculator Notes 3B** and **1B.** ◀] Then record Walker B's motion, and download these data to the same graphing calculator. Answers will vary.

Step 5 Answers will vary, but equations should be approximately $y = 0.5 + x$ for Walker A and $y = 2 + 0.5x$ for Walker B, where x represents time in seconds and y represents the position on the marked line in meters.

Next you'll model the walks with a system of equations.

Step 5 Find an equation to model the data for each of the two walkers.

Step 6 Graph the two equations on the same set of axes with both sets of data. Find the approximate point where the lines intersect.

Step 7 Explain the real-world meaning of the intersection point in Step 6.

Step 8 Check that the coordinates of the point of intersection satisfy both of your equations.

Step 7 After 3 s, both walkers are at the 3.5 m mark.

Step 8 Substitute $x = 3$ and $y = 3.5$ into both equations:

Walker A	Walker B
$3.5 = 0.5 + 3$	$3.5 = 2 + 0.5(3)$

Next you'll consider what happens under different conditions.

Step 9 Suppose that Walker A walks faster than 1 m/s. How is the graph different? What happens to the point of intersection?

Step 10 Suppose that two people walk at the same speed and direction from different starting marks. What does this graph look like? What happens to the solution point?

Step 11 Suppose that two people walk at the same speed in the same direction from the same starting mark. What does this graph look like? How many points satisfy this system of equations? The two lines overlap on the graph. Every point on the line is a solution to the system of equations.

Step 9 The line for Walker A will be steeper, and the solution point will move toward the y-axis.

Step 10 This graph shows two parallel lines. There is no solution point because the lines never meet.

In the investigation you were asked to find the point of intersection of two lines. In this example you'll see how you can find or confirm a point of intersection using a graph, a table of values, and some calculations.

EXAMPLE

Edna leaves the trailhead at dawn to hike 12 mi toward the lake, where her friend Maria is camping. At the same time, Maria starts her hike toward the trailhead. Edna is walking uphill so she averages only 1.5 mi/h, while Maria averages 2.5 mi/h walking downhill. When and where will they meet?

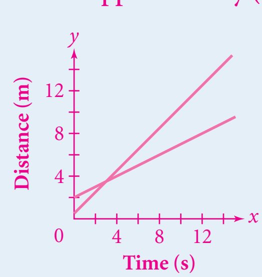
a. Define variables for time and for distance from the trailhead.

b. Write a system of two equations to model this situation.

c. Solve this system by creating a table and finding the values for the variables that make both equations true. Then locate this solution on a graph.

d. Check your solution and explain its real-world meaning.

Step 6 Graphs will vary. The point of intersection should be approximately (3, 3.5).



Step 10 Encourage a variety of starting points.

Step 11 You might also ask students to consider a scenario in which two people walk in opposite directions.

▶ ***Solution***

Both women hike the same amount of time. When Edna and Maria meet they will both be the same distance from the trailhead, although they will have hiked different distances.

a. Let x represent the time in hours. Let y represent the distance in miles from the trailhead.

b. The system of equations that models this situation is grouped in a brace:

$$\begin{cases} y = 1.5x & \text{Edna's hike.} \\ y = 12 - 2.5x & \text{Maria's hike.} \end{cases}$$

Edna starts at the trailhead so she increases her distance from it as she hikes 1.5 mi/h for x hours. Maria starts 12 mi from the trailhead and reduces her distance from it as she hikes 2.5 mi/h for x hours.

c. Create a table from the equations. Fill in the times and calculate each distance. The table shows the x-value that gives equal y-values for both equations. When $x = 3$, both y-values are 4.5. So the solution is the ordered pair (3, 4.5). We say that these values "satisfy" both equations.

Hiking Times and Distances

x	$y = 1.5x$	$y = 12 - 2.5x$
0	0	12
1	1.5	9.5
2	3	7
3	4.5	4.5
4	6	2
5	7.5	−0.5

SHARING IDEAS

Have students present different responses to Steps 7, 9, 10, and 11. Ask why there is variety. They might mention the need to accelerate from standing still and the difficulty of maintaining a constant speed.

Talk through the logic of verifying answers, as in Step 8. Be sure students do not get the notion that they can just substitute and then manipulate the equation to get one that is true. Instead, help students see the validity of evaluating the expressions on each side of the equal sign and showing that they are equal.

[Ask] "What are the disadvantages of the graphical method of solving a system of equations?" To motivate the rest of the chapter, elicit the ideas that the result might be only approximate and that the problem solver needs to construct graphs rather than just work with the equations.

EXAMPLE

This example shows students how to solve a system of linear equations using a calculator graph or table.

You might draw a diagram or ask two students to act out the situation to help students envision what's happening. **[Ask]** "Does Edna's graph slope upward because she's hiking uphill?" [No; stress that the graph rises because her distance from the trailhead increases with time.]

In part b of the solution, Maria's equation is written in intercept form. Edna's equation is a direct variation (also in intercept form with a y-intercept of 0).

Encourage thinking about different ways to approach a problem. If a student asks, "Why not just divide the 12 mi by the speed of 4 mi/h at which the two hikers are approaching each other?" point out that if the problem can be solved that way,

the result is the same as if Maria didn't hike and Edna walked (or jogged) at 4 mi/h. **[Ask]** "Are the situations the same?" [This would give the time they meet, but not the place.]

In part c of the solution, you may need to point out that the pair (x, y) is the solution. Also note the use of the word *satisfy.* It's used later, too.

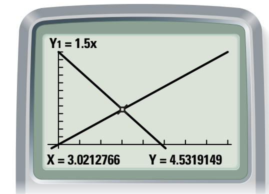
For another example, you could use the CBL 2 demonstration Hot and Cold, in which data gathered from temperature probes are graphed to create a roughly linear system of equations.

Verifying Solutions
Each solution to a system should be checked in both of the original equations, and the check must be logically correct. A solution can be checked by evaluating both sides of the equation and showing that they are equal.

One common faulty method involves substituting into only one of the equations and not discovering that the solution does not satisfy the other equation. Another common mistake is making an error in solving for a variable and then checking in the incorrect form of the equation. Make sure that students check their solution in the two original equations to avoid these pitfalls.

Assessing Progress
You can assess understanding of time-distance relationships and graphs and of how slopes of lines represent rates of change. You can also observe skill at plotting points, writing equations to represent motion with constant speed, and graphing linear equations.

On the graph this solution is the point where the two lines intersect. You can use the trace function on your calculator to approximate the coordinates of the solution point, though sometimes you'll get an exact answer.



[−1, 8, 1, −2, 14, 1]

d. The coordinates (3, 4.5) must satisfy both equations.

Edna	Maria	
$y = 1.5x$	$y = 12 - 2.5x$	Original equations.
$4.5 \stackrel{?}{=} 1.5(3)$	$4.5 \stackrel{?}{=} 12 - 2.5(3)$	Substitute 3 for the time x and 4.5 for the distance y into both equations.
$4.5 = 4.5$	$4.5 = 4.5$	These are true statements, so (3, 4.5) is a solution for both equations.

So, after hiking for 3 h, Edna and Maria meet on the trail 4.5 mi from the trailhead.

Is it possible to draw two lines that intersect in two points? How many possible solutions do you think a linear system of two equations in two variables can have?

When you solve a system of two equations, you're finding a solution in the form (x, y) that makes both equations true. When you have a graph of two distinct linear equations, the solution of the system is the point where the two lines intersect, if they cross at all. You can estimate these coordinates by tracing on the graph. To find the solution more precisely, zoom in on a table. In the next lesson you'll learn how to find the *exact* coordinates of the solution by working with the equations.

Dancers step between the parallel and intersecting sticks of a bamboo dance in Thailand.

EXERCISES

You will need your graphing calculator for Exercises **3, 4, 6, 7,** and **10.**

Practice Your Skills

1. Verify whether the given ordered pair is a solution to the system. If it is not a solution, explain why not.

a. (−15.6, 0.2)
$\begin{cases} y = 47 + 3x \\ y = 8 + 0.5x \end{cases}$

b. (−4, 23)
$\begin{cases} y = 15 - 2x \\ y = 12 + x \end{cases}$

c. (2, 12.3)
$\begin{cases} y = 4.5 + 5x \\ y = 2.3 + 5x \end{cases}$ @

Closing the Lesson

As needed, explain that some problem situations are modeled with a **system of equations.** Solutions to systems of two linear equations can be approximated by seeing where the equations' graphs intersect or by making tables.

1a. yes, because $47 + 3(-15.6) = 0.2$ and $8 + 0.5(-15.6) = 0.2$

1b. No, because $23 \neq 12 + (-4)$; the point satisfies only one of the equations.

1c. No, because $12.3 \neq 4.5 + 5(2)$; furthermore, the lines are parallel, so the system has no solution.

2. Match each graph of a system of equations with its corresponding table values. The tick marks on each graph are one unit apart.

Graph of system

table iv **a.**

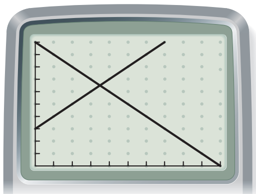


table iii **b.**

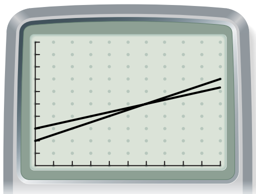


table i **c.**

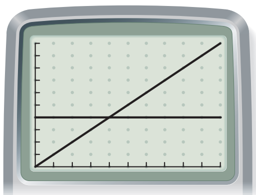


table ii **d.**

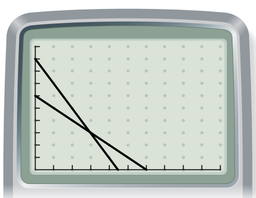


Table values of system

i.

X	Y1	Y2
1	4	1
2	4	2
3	4	3
4	4	4
5	4	5
6	4	6
7	4	7

X = 4

ii.

X	Y1	Y2
−2	13	8
−1	11	7
0	9	6
1	7	5
2	5	4
3	3	3
4	1	2

X = 3

iii.

X	Y1	Y2
3	3.5	4
4	4	4.3333
5	4.5	4.6667
6	5	5
7	5.5	5.3333
8	6	5.6667
9	6.5	6

X = 6

iv.

X	Y1	Y2
2	8	5
2.5	7.5	5.5
3	7	6
3.5	6.5	6.5
4	6	7
4.5	5.5	7.5
5	5	8

X = 3.5

3. Graph each system on your calculator using the window given. Use the trace function to find the point of intersection. Is the calculator giving you approximate or exact solutions?

a. [−18.8, 18.8, 5, −12.4, 12.4, 5]

$$\begin{cases} y = 3 + 0.5x \\ y = -9 + 2x \end{cases}$$ @

b. [−4.7, 4.7, 1, −3.1, 3.1, 1]

$$\begin{cases} y = 4x - 5.5 \\ y = -3x + 5 \end{cases}$$

4. Use the calculator table function to find the solution to each system of equations. (In 4b, you'll need to solve the equations for y first.)

a. $y = 7 + 2.5x$
$y = 35.9 - 6x$

b. $2x + y = 9$
$3x + y = 16.3$

5. Solve the equations for y, then find the value of y when $x = 1$. Substitute these values for x and y into their original equations. What does this tell you?

a. $4x + 2y = 6$

b. $2x - 5y = 20$ @

BUILDING UNDERSTANDING

Students find and solve systems of equations that model situations other than those involving time and distance.

ASSIGNING HOMEWORK

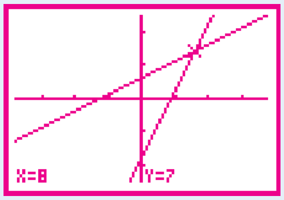
Essential	1–4, 6, 11
Performance assessment	8
Portfolio	8, 10
Journal	7, 11
Group	5, 6, 9
Review	12–16

Helping with the Exercises

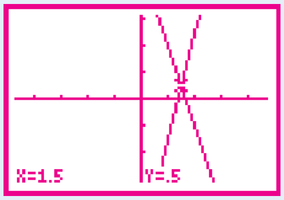
Exercise 1c The lines graphing the system are parallel, so the system has no solution. The given point does lie on one of the lines, though, so accept an answer that says the point isn't a solution because it doesn't lie on the other line.

Exercise 3 Be prepared to explain the list of numbers in the window settings. Visual learners might be helped with a display that shows [Xmin, Xmax, Xscl, Ymin, Ymax, Yscl]. These settings enable the calculator to provide exact solutions for these problems. Students will get approximate solutions if they graph in the standard viewing window (ZOOM 6: ZStandard). Friendly windows are introduced in Chapter 7. (See Calculator Note 7D.)

3a. (8, 7)



3b. (1.5, 0.5)



In this case, the calculator gives exact solutions that satisfy each system.

4a. (3.4, 15.5)

X	Y1	Y2
3	14.5	17.9
3.1	14.75	17.3
3.2	15	16.7
3.3	15.25	16.1
3.4	15.5	15.5
3.5	15.75	14.9
3.6	16	14.3

X=3.4

4b. (7.3, −5.6)

X	Y1	Y2
7	-5	-4.7
7.1	-5.2	-5
7.2	-5.4	-5.3
7.3	-5.6	-5.6
7.4	-5.8	-5.9
7.5	-6	-6.2
7.6	-6.2	-6.5

X=7.3

5a. $y = 3 - 2x$; (1, 1): $4(1) + 2(1) = 6$

5b. $y = -4 + 0.4x$; (1, −3.6): $2(1) - 5(-3.6) = 20$

The point satisfies both forms of the linear equation.

Reason and Apply

6. APPLICATION Two friends start rival Internet companies in their homes. It costs Gizmo.kom \$12,000 to set up the computers and buy the necessary office supplies. Advertisers pay Gizmo.kom \$2.50 for each hit (each visit to the website).

a. Define variables and write an equation to describe the profits for Gizmo.kom. @

b. The profit equation for the rival company, Widget.kom, is $P = -5000 + 1.6N$. Explain possible real-world meanings of the numbers and variables in this equation, and tell why they're different from those in 6a. @

c. Use a calculator table to find the N-value that gives approximately equal P-values for both equations. @

d. Use your answer to 6c to select a viewing window, and graph both equations to display their intersection and all x- and y-intercepts.

e. What are the coordinates of the intersection point of the two graphs? Explain how you found this point and how accurate you think it is.

f. What is the real-world meaning of these coordinates?

7. APPLICATION After seeing her friends profit from their websites in Exercise 6, Sally wants to start a third company, Gadget.kom, with the start-up costs of Widget.kom and the advertising rate of Gizmo.kom.

a. What is Sally's profit equation? $P = -5000 + 2.5N$

b. Graph the profit equations for Gadget.kom and Gizmo.kom.

c. What does the graph tell you about Sally's profits compared to Gizmo.kom's? h

d. What is the x-intercept for Sally's equation? What is its real-world meaning?

8. APPLICATION The total tuition for students at University College and State College consists of student fees plus costs per credit. Some classes have different credit values. The table shows the total tuition for programs with different numbers of credits at each college.

Total Tuition

Credits	University College (\$)	State College (\$)
1	55	47
3	115	111
6	205	207
9	295	303
10	325	335
12	385	399

a. Write a system of equations that represents the relationship between credit hours and total tuition for each college. @

b. Find the solution to this system of equations and check it. @

c. Which method did you use to solve this system? Why?

d. What is the real-world meaning of the solution? @

e. When is it cheaper to attend University College? State College?

Exercise 6 **[Language]** The *start-up costs* for a business are the amount of money that must be spent to begin the business. The *k* in *.kom* is a deliberate misspelling to avoid confusion with actual "dot com" companies. You may need to explain that profit is the difference between revenue (\$2.50 multiplied by the number of hits) and cost (the start-up cost). **[Alert]** Some students may be uncomfortable with the use of variables other than x and y, especially for graphing. Help them convert to x and y. Some students also may need help in setting an appropriate viewing window for large numbers.

6a. Let P represent profit in dollars and N represent the number of hits; $P = -12{,}000 + 2.5N$.

6b. P represents profit, N represents hits. Widget.kom's start-up costs are \$5,000, and its advertisers pay \$1.60 per hit. Because Widget.kom spent less in start-up costs, its website might be less attractive to advertisers, hence the lower rate.

6c. When $N \approx 7778$, $P \approx 7445$ in both equations.

6e. Use the table to find (7778, 7445). Tracing on this graph is not precise.

6f. This intersection point indicates that for 7778 hits to their websites, the two companies make a profit of about \$7,445.

Exercise 7 **[Alert]** If students didn't hear the term *start-up costs* in connection with Exercise 6, they may not know what it means.

7c. Sally will always profit more than Gizmo.kom for the same number of website hits. Because their lines never intersect, there is no solution to the system of equations, and their profits will never be equal.

7d. 2000 hits; after 2000 hits, Sally will have earned back her start-up costs.

See page 723 for answers to Exercises 6d and 7b.

Exercise 8 Some students may find it easier to find the y-intercept if they insert a row for 0 credits into the table. You can use the Sketchpad demonstration Solving Systems of Equations to replace this exercise.

8a. $y = 25 + 30x$, where y is tuition for x credits at University College; $y = 15 + 32x$, where y is tuition for x credits at State College

8b. (5, 175); check: $175 = 25 + 30(5)$, $175 = 15 + 32(5)$

8c. Answers will vary. The table is more accurate than tracing on the calculator graph.

8d. When a student takes 5 credit hours, the tuition at either college is \$175.

8e. It is cheaper to attend University if taking more than 5 credits; for fewer than 5 credits, it is cheaper to attend State. For 5 credits, they cost the same.

9. The high school band and drill team both practice on the football field. During one part of the routine, a drill team member marches from the 9 yd mark on the sideline at 1 yd/s toward the 0-yard mark. At the same time, the tuba player marches from the 3 yd mark at 0.5 yd/s in the opposite direction.

a. Write a system of equations to describe this situation.

b. Find the solution to this system and explain its meaning.
(4, 5); after 4 s, the tuba player bumps into the drill team member at the 5 yd mark.

The marching band performs at halftime during a football game at West Point.

10. The equations $y = 28.65 - 0.0411(x - 1962)$ and $y = 27.5 - 0.0411(x - 1990)$ both model the data for the winning times for the Olympic men's 10,000-meter race. The variable x represents the year, and y represents the winning time, in minutes.

a. Find the approximate winning time for the year 1972 given by each equation. What is the difference between the values?

b. Find the approximate winning time for the year 2008 given by each equation. What is the difference between the values?

c. Select an appropriate window and graph the two equations.

d. Do you think these equations represent the same line? Explain your reasoning. @

11. ***Mini-Investigation*** Consider the system of equations

$$\begin{cases} y = a + bx \\ y = 2 - 5x \end{cases}$$

Explain what values of a and b give this system

a. exactly 1 solution **b.** no solutions **c.** infinitely many solutions @

Review

2.3 **12.** **APPLICATION** Hydroplanes are boats that move so fast they skim the top of the water. The hydroplane *Spirit of the Tri-Cities* qualified for the 2004 Columbia Cup race with a speed of 145.000 mi/h. The hydroplane *Miss B* qualified with a speed of 163.162 mi/h. (Northwest Hydro Racing, *www.hydroracing.com*)

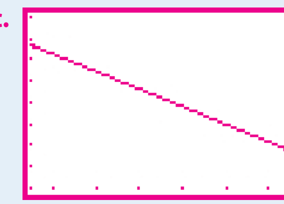
a. How long will each hydroplane take to run a 5-lap race if one lap is 2.5 miles?
Spirit of the Tri-Cities: 5.172 min; *Miss B:* 4.597 min

9a. $d = 9 - t$, where d is the drill team member's distance from the end zone; $d = 3 + 0.5t$, where d is the tuba player's distance from the end zone

10a. The equations give winning times of 28.239 min and 28.2398 min; the difference is 0.0008.

10b. The equations give winning times of 26.7594 min and 26.7602 min; the difference is 0.0008.

10c.



[1945, 2005, 10, 26, 30, 0.5]

10d. $\begin{cases} y = 109.2882 - 0.0411x \\ y = 109.289 - 0.0411x \end{cases}$

The graph in 10c appears to show one line; however, the y-values are 0.0008 unit apart. While the two lines are not identical, they are well within the accuracy of the model, so you could say they are the same model.

11a. Because lines with different slopes always intersect, the y-intercept a can equal any number, and b can be any number except -5.

11b. $a \neq 2$ and $b = -5$; same slope, different y-intercept, lines do not intersect

11c. $a = 2$ and $b = -5$; same slope and y-intercept, lines overlap

Exercise 12 Assume that the boats maintain their qualifying speeds throughout the race. This exercise provides an excellent review of the concepts presented in Lesson 2.3 and shows the usefulness of dimensional analysis, but it is complex. If students have difficulty, encourage them to slowly build up the problem to the result in gallons.

$$\frac{\frac{4.3\text{ gal}}{\text{min}} \cdot \frac{60\text{ min}}{\text{h}} \cdot \frac{2.5\text{ mi}}{\text{lap}} \cdot 5\text{ laps}}{\frac{163.162\text{ mi}}{\text{h}}}$$

$$\begin{aligned} \text{gal} &= \frac{\text{gal}}{\text{min}}(\text{min}) \\ &= \frac{\text{gal}}{\text{min}} \cdot \frac{\text{min}}{\text{h}}(\text{h}) \\ &= \frac{\frac{\text{gal}}{\text{min}} \cdot \frac{\text{min}}{\text{h}}(\text{mi})}{\frac{\text{mi}}{\text{h}}} \\ &= \frac{\frac{\text{gal}}{\text{min}} \cdot \frac{\text{min}}{\text{h}} \cdot \frac{\text{mi}}{\text{lap}} \cdot \text{laps}}{\frac{\text{mi}}{\text{h}}} \end{aligned}$$

12b. *Spirit of the Tri-Cities:* 22.241 gal; *Miss B:* 19.766 gal

12c. *Spirit of the Tri-Cities:* 24.167 mi; *Miss B:* 27.194 mi

12d. *Spirit of the Tri-Cities:* 0.562 mpg; *Miss B:* 0.632 mpg

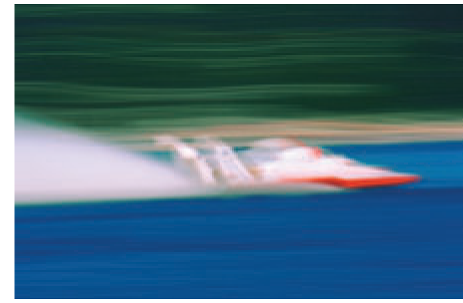
Exercise 13 Be sure the logic of the check by substitution is correct.

Exercise 15 This review of matrices helps prepare for solving systems of equations with matrices in Lesson 5.4. Assign it only if you did Lesson 1.8.

b. Some boats limit the amount of fuel the motor burns to 4.3 gallons per minute. How much fuel will each boat use to run a 5-lap race?

c. Hydroplanes have a 50-gallon tank though generally only about 43 gallons are put in. The rest of the tank is filled with foam to prevent sloshing. How many miles can each hydroplane go on one 43-gallon tank of fuel?

d. Find each boat's fuel efficiency rate in miles per gallon.



This hydroplane travels so fast that its image is blurred in the photo. Learn about hydroplane racing at www.keymath.com/DA .

3.6 **13.** Solve each equation using the method you like best. Then substitute your value for x back into the equation to check your solution.

a. $0.75x = 63.75$ $x = 85$

b. $18.86 = -2.3x$ $x = -8.2$

c. $6 = 12 - 2x$ $x = 3$

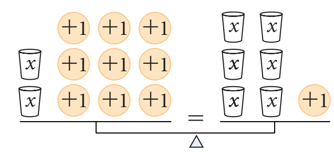
d. $9 = 6(x - 2)$ $x = 3.5$

e. $4(x + 5) - 8 = 18$ $x = 1.5$

14.

$2x + 9 = 6x + 1$	Original equation.
$2x - 2x + 9 = 6x - 2x + 1$	Subtract $2x$ from both sides.
$9 = 4x + 1$	Combine like terms.
$9 - 1 = 4x + 1 - 1$	Subtract 1 from both sides.
$8 = 4x$	Combine like terms.
$\frac{8}{4} = \frac{4x}{4}$	Divide both sides by 4.
$x = 2$	Reduce.

3.6 **14.** Write the equation represented by this balance. Then solve the equation for x using the balancing method. @



1.8 **15.** Find each matrix sum and difference.

a. $\begin{bmatrix} 3 & -3 \\ -9 & 1 \end{bmatrix} + \begin{bmatrix} -2 & -8 \\ 3 & 7 \end{bmatrix}$ $\begin{bmatrix} 1 & -11 \\ -6 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 5 & 0 \\ 2 & 7 \end{bmatrix} - \begin{bmatrix} -8 & 1 \\ -5 & -1 \end{bmatrix}$ $\begin{bmatrix} 13 & -1 \\ 7 & 8 \end{bmatrix}$

4.3 **16.** Solve each equation for y.

a. $y + 2 = 5x$ $y = 5x - 2$

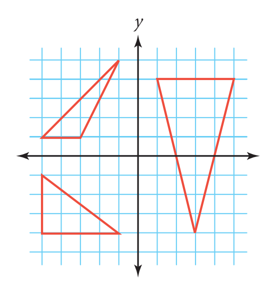
b. $5y = 4 - 7x$ $y = 0.8 - 1.4x$

c. $2y - 6x = 3$ $y = 1.5 + 3x$

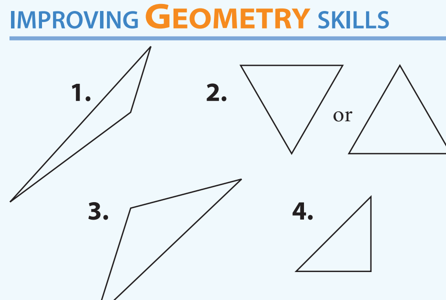
IMPROVING YOUR GEOMETRY SKILLS

Draw a triangle that satisfies each of these sets of conditions. If it's not possible, tell why not.

1. a triangle with all three sides having positive slope
2. an equilateral triangle (three equal sides) with one side having slope 0
3. an isosceles triangle (two equal sides) with all three sides having positive slope
4. a right triangle with one side having undefined slope, one side having slope 0, and one side having slope 1
5. a triangle with two sides having the same slope



IMPROVING GEOMETRY SKILLS



5. This is not possible because two lines with the same slope are parallel and never meet, but each pair of a triangle's edges must meet at a vertex.

LESSON

5.2

Solving Systems of Equations Using Substitution

Graphing systems and comparing their table values are good ways to see solutions. However, it's not always easy to find a good graphing window or the right settings for a table. Also, the solutions you find are often only approximations. To find exact solutions, you'll need to work algebraically with the equations themselves. One way is called the **substitution method.**

EXAMPLE A

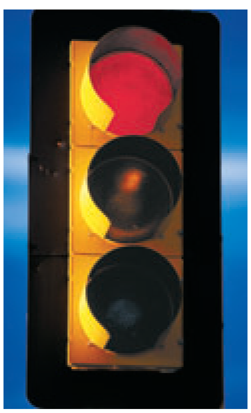
On a rural highway a police officer sees a motorist run a red light at 50 mi/h and begins pursuit. At the instant the police officer passes through the intersection at 60 mi/h, the motorist is 0.2 mi down the road. When and where will the officer catch up to the motorist?

a. Write a system of equations in two variables to model this situation.

b. Solve this system by the substitution method, and check the solution.

c. Explain the real-world meaning of the solution.

▸ *Solution*



Let t represent the time in hours, with $t = 0$ being the instant the officer passes through the intersection. Let d represent the distance in miles from the intersection.

a. The system of equations is

$$\begin{cases} d = 0.2 + 50t \\ d = 60t \end{cases}$$

The first equation represents the motorist, who is already 0.2 mi away when the timing begins. The second equation represents the officer.

b. When the officer catches up to the motorist, they will both be the same distance from the intersection. At this time, both equations will have the same d-value. So you can replace d in one equation with an equivalent expression for d that you find from the other equation. Substituting $60t$ for d into $d = 0.2 + 50t$ gives the new equation:

$$\begin{cases} d = 0.2 + 50t \\ d = 60t \end{cases} \longrightarrow 60t = 0.2 + 50t$$

There is now one equation to solve. Notice that the variable t occurs on both sides of the equal sign and that d has dropped out. Now you use the balancing method to find the solution.

$60t = 0.2 + 50t$	New equation.
$60t - 50t = 0.2 + 50t - 50t$	Subtract $50t$ from both sides of the equation.
$10t = 0.2$	Combine like terms.
$t = 0.02$	Divide both sides of the equation by 10 and reduce.

LESSON

5.2

PLANNING

LESSON OUTLINE

First day:

15 min Example A

20 min Investigation

10 min Sharing

5 min Exercises

Second day:

15 min Example B

5 min Closing

30 min Exercises

MATERIALS

- ropes of different thickness, each about 1 m long, with ends sealed to prevent fraying (two per group)
- meterstick or tape measure
- 9 m thin rope, *optional*
- 10 m thick rope, *optional*
- Rope Sample Data (W), *optional*
- Fathom demonstration Olympic Times, *optional*

TEACHING

The substitution method can provide exact solutions to a system of linear equations. These solutions are not approximations, as is often true for solutions from tables or graphs.

EXAMPLE A

This example introduces the substitution method for two equations in intercept form. You might have students make a diagram or act out the situation. Ask whether there is necessarily a solution to the problem. Because the patrol car is traveling faster than the motorist, a solution will exist. In fact, because the distance between them is 0.2 mi and is closing at 10 mi/h, some students might just

NCTM STANDARDS

CONTENT		PROCESS	
	Number	✓	Problem Solving
✓	Algebra	✓	Reasoning
✓	Geometry	✓	Communication
✓	Measurement	✓	Connections
✓	Data/Probability	✓	Representation

LESSON OBJECTIVES

- Understand the limitations of solving systems graphically
- Solve systems of linear equations using substitution

point out that the time required will be $\frac{0.2}{10}$, or 0.02 h.

[Alert] If students suggest substituting into the same equation, try it to show that all variables will drop out. They must use the other equation.

Some spatially challenged students may understand the substitution better if the two equations are written side by side rather than one under the other.

Guiding the Investigation

One Step
Ask students to tie knots in the ropes and collect data as in Lesson 3.7. Then ask them to find the number of knots that make the thick and thin ropes the same length without using graphs or tables. As you observe, encourage them to write the equations next to each other to see how to substitute.

Steps 1–4 As an alternative to students' collecting data in Steps 1 and 3, you might pass out sample data. Assign each group the data from any one of these rope pairs: 1 and 4; 1 and 5; 2 and 4; 2 and 5; 3 and 4; 3 and 5. To move directly to Step 2, you can give them these equations, based on the sample data:

Type 1 rope: $y = 89.9 - 4.2x$

Type 2 rope: $y = 93.9 - 4.2x$

Type 3 rope: $y = 100 - 6x$

Type 4 rope: $y = 100 - 10.3x$

Type 5 rope: $y = 97.8 - 13.6x$

The units are centimeters.

Step 1 Data will vary; a sample for the thin rope (Type 3):

Number of knots	Length (cm)
0	100
1	94
2	88
3	81.3
4	75.7
5	69.9
6	63.5

To find the d-value of the solution, substitute 0.02 for t into one of the original equations.

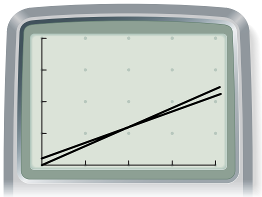
$t = (0.02)$

$$d = 0.2 + 50t \qquad \text{and} \qquad d = 60t$$
$$d = 0.2 + 50(0.02) \qquad\qquad d = 60(0.02)$$
$$d = 1.2 \qquad\qquad d = 1.2$$

If both equations give the same d-value, 1.2 in this case, then you have the correct solution.

c. The solution is the only ordered pair of values, (0.02, 1.2), that works in both equations. The police officer will catch up to the motorist 1.2 mi from the intersection in 0.02 h, which is 1 min 12 s after passing through the intersection.

The calculator screen shows the system of equations from the example in the window [0, 0.04, 0.01, 0, 4, 1]. It is difficult to guess the solution at these window settings because the two lines have very similar slopes and close y-intercepts. But the substitution method helps you find the exact solution no matter how difficult it is to set windows or tables. Once you have the exact solution, it is much easier to find a good window to display it.

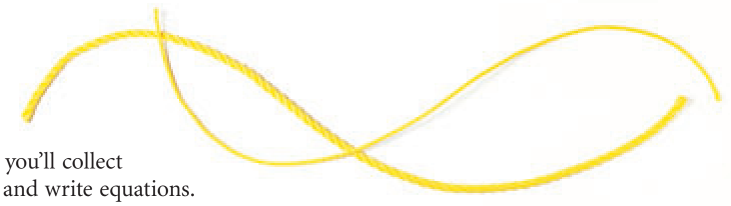


Investigation
All Tied Up

You will need

- two ropes of different thickness, both about 1 meter long
- a meterstick or tape measure
- a 9-meter-long thin rope (optional)
- a 10-meter-long thick rope (optional)

In this investigation you'll work with rope lengths and predict how many knots it would take in each rope to make a thicker rope the same length as a thinner one.



First you'll collect data and write equations.

Step 1 Measure the length of the thinner rope without any knots. Then tie a knot and measure the length of the rope again. Continue tying knots until no more can be tied. Knots should be of the same kind, size, and tightness. Record the data for number of knots and length of rope in a table.

Step 2 Define variables and write an equation in intercept form to model the data you collected in Step 1. What are the slope and y-intercept, and how do they relate to the rope?

Step 2 If x represents the number of knots and y represents the sample-rope length in centimeters, then the equation is $y = 100 - 6x$. The y-intercept, 100, is the length of rope in centimeters without knots, and the slope, -6, is the amount of thin rope in centimeters that each knot takes.

Step 3 Repeat Steps 1 and 2 for the thicker rope.

Step 4 Suppose you have a 9-meter-long thin rope and a 10-meter-long thick rope. Write a system of equations that gives the length of each rope depending on the number of knots tied. A sample system is $\begin{cases} y = 900 - 6x \\ y = 1000 - 10.3x \end{cases}$.

Step 3 Data will vary; a sample for Type 4:

Number of knots	Length (cm)
0	100
1	89.7
2	78.7
3	68.6
4	57.4
5	47.8
6	38.1

If x represents the number of knots and y represents the sample-rope length in centimeters, then the equation is $y = 100 - 10.3x$. The y-intercept, 100, is the length of rope in centimeters without knots, and the slope, -10.3, is the amount of thick rope in centimeters that each knot takes.

Step 4 Equations will vary because they use student-collected data.

Step 6 Possible window is $[0, 40, 5, 0, 1100, 100]$; estimated coordinates from sample: $(23, 760)$.

Step 7 At 23 knots, both ropes have nearly equal lengths of 760 cm.

Next you'll analyze the system to find a meaningful solution.

Step 5 Solve this system of equations using the substitution method.

Step 6 Select an appropriate window setting and graph this system of equations. Estimate coordinates for the point of intersection to check your solution. Compare this solution with the one from Step 5.

Step 7 Explain the real-world meaning of the solution to the system of equations.

Step 8 What happens to the graph of the system if the two ropes have the same thickness? The same length? If two ropes have the same thickness, the slopes will be equal; if they have the same length, the y-intercepts are the same.

So far in this chapter, you've seen equations only in intercept form. In other words, they are already solved for the output variable, y. This form makes it easy to use the substitution method: You can simply set the two expressions in x equal to each other because they are both equal to y. Sometimes you have to put the equations into intercept form before substituting. In the next example, you'll have to change an equation in standard form to intercept form.

EXAMPLE B

A pharmacist has 5% saline (salt) solution and 20% saline solution. How much of each solution should be combined to create a bottle of 90 mL of 10% saline solution?

a. Write a system of equations that models this situation.

b. Solve one equation for x or y and substitute into the other equation to find a solution.

c. Check your solution.

▶ Solution

First decide what your variables are. You are trying to find how much of 5% and how much of 20% saline solution to use. So let x = amount of 5% saline solution, and let y = amount of 20% saline solution, in mL.

a. The total amount of saline solution needed is 90 mL, so write the equation

$$x + y = 90$$

The amount of salt in x mL of 5% saline solution is $0.05x$, and the amount of salt in y mL of 20% saline solution is $0.2y$. The total combined salt must be 10% of 90 mL, or $0.1(90)$. So write the equation

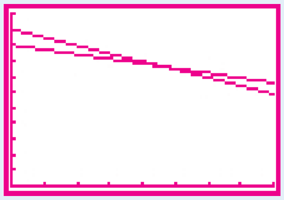
$$0.05x + 0.2y = 0.1(90)$$

So the system of equations that models this situation is

$$\begin{cases} x + y = 90 \\ 0.05x + 0.2y = 9 \end{cases}$$

Step 5 From sample: $900 - 6x = 1000 - 10.3x$, $4.3x = 100$, $x = 23\frac{11}{43}$; round to 23 knots; $y = 760\frac{20}{43}$; round to 760 cm.

Step 6



Step 7 In the solution, x will be the number of knots needed in each rope for the two ropes to be the same length, and y will be that length. Note that the number of knots must be an integer value. Students will need to round off their actual answers to get realistic values.

You may want to check the students' solutions as a class by having each half of the class tie knots in the optional longer ropes and then comparing the ropes. The longer ropes must have the same thickness as the shorter ropes.

You might have students write a lab report for this investigation.

SHARING IDEAS

Have students present several different systems for Steps 4 and 5. Ask students why the results differ. Then lead a discussion of ideas about Step 8. **[Ask]** "Solve the system $3x - 4y = 11$ and $3x + 2y = -1$ in as many ways as you can." Not only does this get students thinking about solving equations in standard form, but it also motivates solving by elimination, the topic of Lesson 5.3.

Assessing Progress

Watch for the ability to collect data systematically into tables, write linear equations in intercept form, find the slope and y-intercept, and decide the conditions under which a system has zero solutions, one solution, or infinitely many solutions.

EXAMPLE B

Ask students to predict the answer before solving the equations. They can better judge the correctness of their results if they first decide on a reasonable estimate. Students may reason that 10% is closer to 5% than 20% is, so they will need more of the 5% solution. Make sure that students understand that the first equation, $x + y = 90$, represents the total amount of saline solution, while the second equation, $0.05x + 0.2y = 0.1(90)$, represents the amount of salt in the solution. Exercises 13 and 14 are mixture problems.

The More Practice Your Skills worksheet for Lesson 5.7 gives more practice with rate, work, and mixture problems.

Closing the Lesson

The **substitution method** allows you to solve a system of linear equations exactly and without graphing or constructing a table. Following are the stages of this method:

1. Solve both equations for one variable and set the two equations equal. Or solve one equation for one variable and substitute that value into the other equation.

2. Solve the resulting equation for the other variable.

3. Substitute that solution for that variable into the first equation.

4. Solve for the second variable.

b. To use the substitution method, one of the equations must be solved for the variable. It'll be easiest to solve the first equation for one of the variables. You can solve for either x or y, using the balancing method.

$x + y = 90$	Original equation.
$x + y - y = 90 - y$	Subtract y from both sides.
$x = 90 - y$	Combine like terms.

Substitute $90 - y$ for x into the second equation, and solve for y.

$0.05x + 0.2y = 9$	Original equation.
$0.05(90 - y) + 0.2y = 9$	Substitute $90 - y$ for x.
$4.5 - 0.05y + 0.2y = 9$	Distribute the 0.05 through the parentheses.
$4.5 + 0.15y = 9$	Combine like terms.
$0.15y = 4.5$	Subtract 4.5 from both sides.
$y = 30$	Divide both sides by 0.15 and reduce.

To find the corresponding x-value, substitute 30 for y into one of the equations.

$x = 90 - y$	The first equation, in intercept form.
$x = 90 - 30 = 60$	Substitute 30 for y and evaluate.

c. To check your solution, substitute 60 for x and 30 for y into the original equations.

$x + y = 90$	$0.05x + 0.2y = 9$
$30 + 60 \stackrel{?}{=} 90$	$0.05(60) + 0.2(30) \stackrel{?}{=} 9$
$90 = 90$	$3 + 6 \stackrel{?}{=} 9$
	$9 = 9$

Both equations result in true statements, so the solution is correct. So the pharmacist must combine 60 mL of 5% saline solution and 30 mL of 20% saline solution.

Problems like those in Example B are called **mixture problems.** This type of problem often involves a system of equations.

There are many ways to solve systems by using the substitution method. You can set expressions equal to one another, or solve for one of the variables and substitute the expression you get into the other equation. Both ways are examples of **symbolic manipulation,** which simply means that you are working with the properties you have used in the balancing method and "undoing" to keep sides of the equation equal. It does not matter which equation or variable you work with first, but you must always substitute the resulting expression into the *other* equation to find a solution. When you solve a system of equations using the substitution method, you can always find an exact solution, not just its approximate coordinates.

EXERCISES

You will need your graphing calculator for Exercises **12** and **15.**

Practice Your Skills

1. The system of equations

$$\begin{cases} d = 1.5t \\ d = 12 - 2.5t \end{cases}$$

describes the distance of two hikers, Edna and Maria, from the example in Lesson 5.1. By setting the expressions of the right sides of the equations equal to each other, you can find the time when Edna and Maria meet. Explain what happens in Stages 3 and 5 of the substitution process.

$d = 12 - 2.5t$	1. Original equation.
$1.5t = 12 - 2.5t$	2. Substitute $1.5t$ for d.
$1.5t + 2.5t = 12 - 2.5t + 2.5t$	3. Add $2.5t$ to both sides.
$4t = 12$	4. Combine like terms.
$\frac{4t}{4} = \frac{12}{4}$	5. Divide both sides by 4.
$t = 3$	6. Reduce.

2. Check that each ordered pair is a solution to each system. If the pair is not a solution point, explain why not. ⓗ

a. $(-2, 34)$
$$\begin{cases} y = 38 + 2x \\ y = -21 - 0.5x \end{cases}$$

b. $(4.25, 19.25)$
$$\begin{cases} y = 32 - 3x \\ y = 15 + x \end{cases}$$

c. $(2, 12.3)$
$$\begin{cases} y = 2.3 + 3.2x \\ y = 5.9 + 3.2x \end{cases}$$

3. Solve each equation by symbolic manipulation.

a. $14 + 2x = 4 - 3x$ ⓐ

b. $7 - 2y = -3 - y$ ⓐ

c. $5d = 9 + 2d$

d. $12 + t = 4t$

4. Solve the system of equations using the substitution method, and check your solution. ⓗ

$$\begin{cases} y = 25 + 30x \\ y = 15 + 32x \end{cases}$$

(5, 175); check: $175 = 25 + 30(5)$ and $175 = 15 + 32(5)$

5. Substitute $4 - 3x$ for y. Then rewrite each expression in terms of one variable.

a. $5x + 2y$ $\quad 5x + 2(4 - 3x) = 5x + 8 - 6x = -x + 8$

b. $7x - 2y$ ⓐ $\quad 7x - 2(4 - 3x) = 7x - 8 + 6x = 13x - 8$

6. Solve each system of equations by substitution, and check your solution.

a. $\begin{cases} y = 4 - 3x \\ y = 2x - 1 \end{cases}$ (1, 1); check: $1 = 4 - 3(1)$ and $1 = 2(1) - 1$

b. $\begin{cases} 2x - 2y = 4 \\ x + 3y = 1 \end{cases}$ $\left(\frac{7}{4}, -\frac{1}{4}\right)$ or $(1.75, -0.25)$; check: $2(1.75) - 2(-0.25) = 4$ and $1.75 + 3(-0.25) = 1$

2a. no, because the point satisfies only the first equation

2b. yes, because $19.25 = 32 - 3(4.25)$ and $19.25 = 15 + 4.25$

2c. No, because the point satisfies only the second equation; furthermore, the lines have the same slope, so they are parallel and there is no solution.

3a. $2x + 3x = 4 - 14$
$5x = -10$
$x = -2$

3b. $-2y + y = -3 - 7$
$-y = -10$
$y = 10$

3c. $5d - 2d = 9$
$3d = 9$
$d = 3$

3d. $t - 4t = -12$
$-3t = -12$
$t = 4$

BUILDING UNDERSTANDING

Students practice solving systems of equations by substitution.

ASSIGNING HOMEWORK

Essential	1–4, 6, 7, 10
Performance assessment	8, 9, 12
Portfolio	12
Journal	8, 10
Group	5, 11, 13, 14
Review	15–18

Helping with the Exercises

Exercise 2 Be sure students use good logic as they verify solutions by evaluating both sides and comparing them. Even without substituting the value into either equation, students may recognize that both equations in 2c have the same slope, which means the lines are parallel and therefore cannot intersect.

Reason and Apply

7. **APPLICATION** This system of equations models the profits of two home-based Internet companies.

$$\begin{cases} P = -12000 + 2.5N \\ P = -5000 + 1.6N \end{cases}$$

The variable P represents profit in dollars, and N represents hits to the company's website.

a. Use the substitution method to find an exact solution. @ See below.

b. Is an approximate or exact solution more meaningful in this model? @ The approximate solution, $N \approx 7778$ and $P \approx 7444$, is more meaningful because there cannot be a fractional number of website hits.

8. The costs for two families to attend Friday night's basketball game are given by $2x + 3y = 13.50$ and $3x + 2y = 16.50$, where x is the cost of an adult ticket and y is the cost of a student ticket, in dollars.

a. What is the real-world meaning of the first equation?

b. Solve this system of equations using the substitution method.

c. What are the prices of adult and student tickets?

9. **APPLICATION** The manager of a movie theater wants to know the number of adults and children who go to the movies. The theater charges \$8 for each adult ticket and \$4 for each child ticket. At a showing where 200 tickets were sold, the theater collected \$1304.

a. Let the variable A represent the number of adult tickets and C represent the number of child tickets. Write an equation for the total number of tickets sold. @

b. Write an equation showing the total cost of the tickets. @

c. Use your equations from 9a and b to write a system whose solution represents the number of adult and child tickets sold. Solve this system by symbolic manipulation.

10. Students in an algebra class did an experiment similar to the Investigation Where Will They Meet? from Lesson 5.1. They wrote the system

$$\begin{cases} d = 0.5 + 0.75t \\ d = 2.5 + 0.75t \end{cases}$$

a. What real-world information does the system tell you? The first walker starts at the 0.5 m mark and walks away at 0.75 m/s. The second walker starts at the 2.5 m mark and walks away at 0.75 m/s.

b. Use the substitution method to solve this system. no solution

c. What is the real-world meaning of the solution you found in 10b? The walkers will never meet.

8a. The total admission price for two adults and three students is \$13.50.

8b. $x = 4.5$ and $y = 1.5$

8c. An adult ticket costs \$4.50, and a student ticket costs \$1.50.

Exercise 9 This problem is a variation on an old puzzle, so a student might point out that it can be solved without using a system of equations. Each of the 200 tickets brought in \$4, making \$800. The remainder of the ticket sales, \$504, came from the adults, each of whom paid an additional \$4. So there were $\frac{504}{4}$, or 126, adults. Encourage good thinking like this. Challenge these students to represent each step of their reasoning in a system of equations. Their method will motivate solving systems by elimination in Lesson 5.3.

9a. $A + C = 200$

9b. $8A + 4C = 1304$

9c. $A = 126$ and $C = 74$, so the theater sold 126 adult tickets and 74 child tickets.

7a. Answers will vary. A sample solution:

$$\begin{aligned} -12{,}000 + 2.5N &= -5{,}000 + 1.6N && \text{Set equations equal to each other.} \\ -12{,}000 + 0.9N &= -5{,}000 && \text{Subtract } 1.6N \text{ from both sides.} \\ 0.9N &= 7{,}000 && \text{Add 12,000 to both sides.} \\ N &= \frac{70{,}000}{9} = 7{,}777\tfrac{7}{9} && \text{Divide both sides by 0.9.} \\ P &= -12{,}000 + 2.5\left(\frac{70{,}000}{9}\right) = 7{,}444\tfrac{4}{9} \end{aligned}$$

11. The table at right gives the equations that model the three vehicles' distances in the Investigation On the Road Again from Lesson 3.2.

The variable d represents the distance in miles from Flint and t represents time in minutes, with $t = 0$ being the instant all three vehicles start traveling.

For each event described in 11a–c, write a system of equations, solve using the substitution method, and explain the real-world meaning of your solution.

Distance from Flint

Equation	Vehicle
$d = 220 - 1.2t$	minivan
$d = 35 + 0.8t$	sports car
$d = 1.1t$	pickup truck

a. The pickup truck passes the sports car. ⓐ

b. The minivan meets the pickup truck.

c. The minivan meets the sports car.

d. Write and solve an equation to find when the minivan is twice as far from Flint as the sports car. ⓗ $220 - 1.2t = 2(35 + 0.8t)$, $t \approx 53.6$ min; minivan is about 156 mi, sports car is about 78 mi.

12. **APPLICATION** This table shows the winning times for the Olympic women's and men's 100-meter breaststroke. The times are given in minutes and seconds. For example, 1:15.80 means 1 min 15.80 s.

Women's and Men's 100-meter Breaststroke

Year	Women's champion and country	Time	Men's champion and country	Time
1968	Djurdjica Bjedov, Yugoslavia	1:15.80	Donald McKenzie, United States	1:07.70
1972	Catherine Carr, United States	1:13.58	Nobutaka Taguchi, Japan	1:04.94
1976	Hannelore Anke, East Germany	1:11.16	John Hencken, United States	1:03.11
1980	Ute Geweniger, East Germany	1:10.22	Duncan Goodhew, Great Britain	1:03.44
1984	Petra Van Staveren, Netherlands	1:09.88	Steve Lundquist, United States	1:01.65
1988	Tanya Dangalakova, Bulgaria	1:07.95	Adrian Moorhouse, Great Britain	1:02.04
1992	Elena Roudkovskaia, Unified Team	1:08.00	Nelson Diebel, United States	1:01.50
1996	Penny Heyns, South Africa	1:07.73	Frédéric Deburghgraeve, Belgium	1:00.60
2000	Megan Quann, United States	1:07.05	Domenico Fioravanti, Italy	1:00.46
2004	Xuejuan Luo, China	1:06.64	Kosuke Kitajima, Japan	1:00.08

(International Olympics Committee, in *The World Almanac and Book of Facts 2004*, pp. 870, 872) [Data sets: **SWMYR, SWMWM, SWMMN**]

a. Find a line of fit based on Q-points for the women's and the men's data sets. (*Hint:* You'll probably want to change the times to seconds. For example, 1:15.80 is 75.80 s.) ⓐ

b. Solve a system of equations whose solution tells you when the men and women will have equal winning times for this Olympic event. ⓐ $x \approx 2238$, $y \approx 26.23$

c. Select an appropriate window to graph this system and its solution.

d. Discuss the reasonableness of this model and the solution. ⓐ

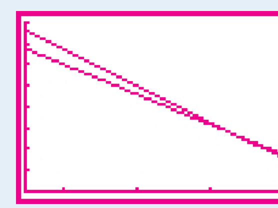
Japan's Kosuke Kitajima swims to win the men's 100-meter breaststroke final at the Athens 2004 Olympic Games.

Exercise 11d Students may be confused by the phrase "twice as far." Ask them how far the sports car is from Flint at any time, and then ask what twice that distance would be.

Exercise 12 The Fathom demonstration Olympic Times can replace this exercise.

12a. women: $y = 71.16 - 0.1715(x - 1976)$ or $y = 67.73 - 0.1715(x - 1996)$; men: $y = 63.44 - 0.142(x - 1976)$ or $y = 60.60 - 0.142(x - 1996)$

12c. Answers will vary. The window shown is [1950, 2300, 100, 0, 80, 10].



12d. The solution means that in the year 2238 (a little more than 230 years from now), both men and women will swim this race in 26.23 s. This is not likely. The model may be a good fit for the data, but extrapolating that far into the future produces unlikely predictions.

11a. $\begin{cases} d = 35 + 0.8t \\ d = 1.1t \end{cases}$

$1.1t = 35 + 0.8t$; $\left(116\frac{2}{3}, 128\frac{1}{3}\right)$

The pickup passes the sports car roughly 128 mi from Flint after approximately 117 min.

11b. $\begin{cases} d = 220 - 1.2t \\ d = 1.1t \end{cases}$

$220 - 1.2t = 1.1t$, $\left(\frac{2200}{23}, \frac{2420}{23}\right) \approx (95.7, 105.2)$

The minivan meets the pickup truck about 105 mi from Flint after approximately 96 min.

11c. $\begin{cases} d = 220 - 1.2t \\ d = 35 + 0.8t \end{cases}$

$35 + 0.8t = 220 - 1.2t$; $(92.5, 109)$

The minivan meets the sports car 109 mi from Flint after 92.5 min.

Exercise 13 Exercise 13 is a mixture problem, but students may not see the similarity to Example B. The equations for Example B modeled the total amount of solution and the total amount of salt. In this exercise, students should consider the total amount of candy and the total cost of the candy.

16c. $y = 100 + 12.1x$, where x represents the time in seconds and y represents her height above ground level. To find out how long her ride to the observation deck is, solve the equation $520 = 100 + 12.1x$.

Exercise 18 The return to matrices helps prepare for Lesson 5.4. Assign this problem only if you did Lesson 1.8 and plan to do Lesson 5.4.

13. A candy store manager is making a sour candy mix by combining sour cherry worms, which cost her \$2.50 per pound, and sour lime bugs, which cost her \$3.50 per pound. How much of each candy should she include if she wants 20 pounds of a mix that costs her a total of \$65? @ 5 lb of sour cherry worms and 15 lb of sour lime bugs

14. Mrs. Abdul mixes bottled fruit juice with natural orange soda to make fruit punch for a party. The bottled fruit juice is 65% real juice and the natural orange soda is 5% real juice. How many liters of each are combined to make 10 liters of punch that is 33% real juice? $4\frac{2}{3}$ L of bottled fruit juice and $5\frac{1}{3}$ L of natural orange soda

Review

4.2 **15.** A system of two linear equations has the solution $(3, -4.5)$. Write the equations of

a. A horizontal line through the solution point. $y = -4.5$

b. A vertical line through the solution point. $x = 3$

4.2 **16.** You and your family are visiting Seattle and take the elevator to the observation deck of the Space Needle. The observation deck is 520 ft high while the needle itself is 605 ft high. The elevator travels at a constant speed, and it takes 43 s to travel from the base at 0 ft to the observation deck.

a. What is the slope of the graph of this situation? @ 12.1 ft/s

b. If the elevator could go all the way to the top, how long would it take to get there? @ 50 s

c. If a rider got on the elevator at the restaurant at the 100 ft level, what equation models her ride to the observation deck? @

The Space Needle, shown here in the city skyline, was built for the 1962 Seattle World's Fair. For interesting information about the Space Needle, see the links at www.keymath.com/DA .

0.1 **17.** Do each calculation by hand, and then check your results with a calculator. Express your answers as fractions.

a. $3 - \frac{5}{6}$ $2\frac{1}{6}$

b. $\frac{1}{4} + \frac{5}{12}$ $\frac{2}{3}$

c. $\frac{3}{4} \cdot \frac{2}{9}$ $\frac{1}{6}$

d. $\frac{1}{5} + \frac{2}{3} + \frac{3}{4}$ $\frac{97}{60}$, or $1\frac{37}{60}$

1.8 **18.** Match each matrix multiplication with its answer.

iii **a.** $\begin{bmatrix} 8 & -2 \\ 1 & 9 \end{bmatrix} \times \begin{bmatrix} 3 & 8 \\ -1 & -4 \end{bmatrix}$ **i.** $\begin{bmatrix} 26 & 72 \\ -6 & -28 \end{bmatrix}$

iii **b.** $\begin{bmatrix} 24 & -16 \\ -1 & -36 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **ii.** $\begin{bmatrix} 36 \\ -17 \end{bmatrix}$

ii **c.** $\begin{bmatrix} 6 & 8 \\ -7 & -1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ **iii.** $\begin{bmatrix} 24 & -16 \\ -1 & -36 \end{bmatrix}$

LESSON

5.3

Solving Systems of Equations Using Elimination

I happen to feel that the degree of a person's intelligence is directly reflected by the number of conflicting attitudes she can bring to bear on the same topic.

LISA ALTHER

You have seen how to approximate the solution to a system of equations using a table or graph, and you've seen how to calculate the exact answer to a system of equations using the substitution method. In this lesson you'll learn another method for finding an exact solution, which will have advantages for certain systems.

You know that when you add equal quantities to each side of an equation, the resulting equation is equivalent and has the same solution as the original.

$y - 7 = 12$	Original equations.	$3x - 5y = 9$
$+ \quad 7 = 7$	Add equal quantities to both sides.	$+ \quad 5y = 5y$
$y \quad = 19$	The resulting equations are true and have the same solutions as the originals.	$3x \quad = 9 + 5y$

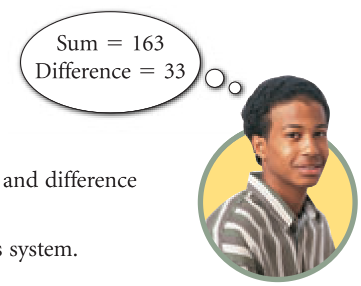
In the same way, when you add two quantities that are equal, c and d, to two other quantities that are equal, a and b, the resulting expressions are equal.

$a = b$	Original equation.
$+\ c = d$	Add equal quantities.
$a + c = b + d$	The resulting equation is true and has the same solutions as the originals.

The **elimination method** makes use of this fact to solve systems of linear equations.

EXAMPLE A

J. P. is thinking of two numbers, but he won't say what they are. He tells you that the sum of the two numbers is 163 and that their difference is 33. Find the two numbers.



a. Write a system of equations for the sum and difference of these numbers.

b. Use the elimination method to solve this system.

▶ Solution

a. Let f and s represent the first and second numbers, respectively. Then the system is

$$\begin{cases} f + s = 163 \\ f - s = 33 \end{cases}$$

The first equation describes the sum, and the second describes the difference.

b. Note that adding the equations eliminates the variable s. Then solve for f.

$f + s = 163$ $+\ f - s = 33$	Original equations.
$2f = 196$	Add.
$f = 98$	Divide both sides by 2.

So the first number is 98. Now you need to find the second number.

LESSON

5.3

PLANNING

LESSON OUTLINE

First day:

10 min	Introduction, Example A
40 min	Investigation

Second day:

10 min	Investigation
10 min	Sharing
10 min	Example B
5 min	Closing
15 min	Exercises

MATERIALS

- paper clips (4 per group)
- pennies (15 per group)
- 8.5-by-11 in. sheets of paper (1 per group)
- Calculator Note 4B

TEACHING

The technique of elimination is useful for solving a system of equations in which no variable has the coefficient 1.

INTRODUCTION

You may want to demonstrate that the pairs of equations in the introduction have the same solutions: 19 for the two equations on the left, and points such as $(3, 0)$ and $\left(4, \frac{3}{5}\right)$ for the equations on the right.

EXAMPLE A

This example introduces the method of elimination. Be sure students understand the meanings of the variables and equations. The plus sign between the two equations emphasizes that you're adding them.

Point out that, although substituting was used to solve for the second variable, the method of

NCTM STANDARDS

CONTENT		PROCESS	
	Number		Problem Solving
✓	Algebra	✓	Reasoning
	Geometry	✓	Communication
✓	Measurement		Connections
	Data/Probability	✓	Representation

LESSON OBJECTIVES

- Solve systems of linear equations in two variables by eliminating one variable
- See more examples in which two linear equations can model real-world situations

substitution isn't being used here because different steps began the process of elimination. The solution to the system would be expressed algebraically as $(f, s) = (98, 65)$.

Be careful to write the check of the solutions in a logically correct way.

Guiding the Investigation

One Step
Ask students to line up four paper clips along the long edge of a piece of paper and to fill in the rest of the length with pennies. Then have them do the same with two paper clips along the short edge and write and solve a system of equations representing the situation. As you circulate, suggest that they double the second equation and combine it somehow with the first to eliminate a variable. Then have them use three instead of four paper clips along the long edge and try to solve the resulting equations by elimination.

Step 1 All groups should use the same size paper clips.

Step 3 Students may need to use clips and pennies that were used along the long edge.

Step 5 **[Ask]** "Why was C eliminated and not P?" [It's easier to eliminate C, but encourage a variety of approaches: multiplying one equation by -2 or dividing the other by -2, for example.]

Step 10 Answers will vary. Regular: (0.75, 1.25); jumbo: (0.75, 2). The diameter of the penny is 0.75 in. The regular paper clip is 1.25 in. long, and the jumbo clip is 2 in. long. Linear systems in experiments like this one can measure walking and running strides, percent mixture in solutions, and city and highway fuel mileage.

To find s, substitute 98 for f into one of the original equations:

$$98 + s = 163 \qquad \text{or} \qquad 98 - s = 33$$

Either way, the second number is 65. Check that your solutions are correct.

$$\begin{aligned} f + s &= 163 \\ 98 + 65 &\stackrel{?}{=} 163 \\ 163 &= 163 \end{aligned} \qquad \begin{aligned} f - s &= 33 \\ 98 - 65 &\stackrel{?}{=} 33 \\ 33 &= 33 \end{aligned}$$

Adding the two equations quickly leads to a solution because the resulting equation has only one variable. The other variable was eliminated! However, you won't always have coefficients that add to 0. In these cases, you'll need another strategy for the elimination method to work.

Investigation
Paper Clips and Pennies

You will need
- three paper clips
- several pennies
- an 8.5-by-11-inch sheet of paper

In this investigation you'll create a system of equations by using paper clips and pennies as variables.

Step 1 Lay one paper clip along the long side of the paper. Then add enough pennies to complete the 11-inch length.

Step 2 Use C for the length of one paper clip and P for the diameter of one penny. Write an equation in standard form showing your results.

Steps 1 and 2 sample answers: for a regular paper clip, $C + 13P = 11$; for a jumbo paper clip, $C + 12P = 11$

Step 3 Now you'll write the other equation for the system. Lay two paper clips along the shorter edge of your paper, and then add pennies to complete the 8.5-inch length.

Step 4 Using the same variables as in Step 2, write an equation to record your results for the shorter side.

Steps 3 and 4 sample answers: for a regular paper clip, $2C + 8P = 8.5$; for a jumbo paper clip, $2C + 6P = 8.5$

Step 5 In this system the equations from Steps 2 and 4 have different coefficients for each variable. What can you do to one equation so that the variable C is eliminated when you add both equations?

Step 6 Use your answer to Step 5 to set up the addition of two equations. Once you eliminate the variable C, use the balancing method to solve for P.

Step 6 The same equation results from either sample system: $-18P = -13.5$; $P = 0.75$. This makes sense because the penny's diameter is constant.

Step 7 Substitute the value for P into one of the original equations to find C.

Step 8 Check that your solution satisfies both equations.

Step 9 Describe at least one other way to solve this system by elimination.

Step 10 Explain the real-world meaning of the solution. Describe other experiments in measuring that you can solve using a system of equations.

Step 9 Descriptions will vary. For example, you can eliminate P first and then substitute to find C. Or, instead of substituting to find the second value, you can start again and eliminate the other variable in the system.

Step 5 In both samples you need to multiply the top equation by (-2).

regular: $\begin{cases} C + 13P = 11 \\ 2C + 8P = 8.5 \end{cases} \rightarrow \begin{cases} -2C - 26P = -22 \\ 2C + 8P = 8.5 \end{cases}$

jumbo: $\begin{cases} C + 12P = 11 \\ 2C + 6P = 8.5 \end{cases} \rightarrow \begin{cases} -2C - 24P = -22 \\ 2C + 6P = 8.5 \end{cases}$

Step 7 regular: $C = 1.25$; jumbo: $C = 2$

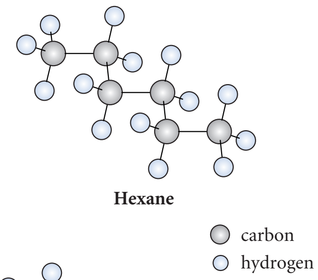
Step 8 regular: $\begin{cases} 1.25 + 13(0.75) = 11 \\ 2(1.25) + 8(0.75) = 8.5 \end{cases}$

jumbo: $\begin{cases} 2 + 12(0.75) = 11 \\ 2(2) + 6(0.75) = 8.5 \end{cases}$

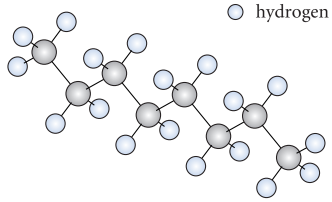
The goal of the elimination method is to get one of the variables to have a coefficient of 0 when you add the two equations. If you start with additive inverses, such as s and $-s$ in Example A, then you can simply add the equations. But often you must first multiply one or both of the equations by some convenient number before you combine them.

EXAMPLE B

A molecule of hexane, C_6H_{14}, has six carbon atoms and fourteen hydrogen atoms. Its molecular weight in grams per mole, the sum of the atomic weights of carbon and hydrogen, is 86.178. The molecular weight of octane, C_8H_{18}, is 114.232 grams per mole. Octane has eight carbon atoms and eighteen hydrogen atoms per molecule. Find the atomic weights of carbon and hydrogen.



Hexane


Octane

a. Define variables and write a system of linear equations in the standard form $ax + by = c$ for these molecular weights.

b. Use elimination to solve this system.

c. Check your solution in the original equations.

▸ Solution

a. Let c represent the atomic weight of carbon in grams per mole. Let h represent the atomic weight of hydrogen in grams per mole. Because the molecular weight of the compounds is the sum of the atomic weights of carbon and hydrogen, you can write the system

$$\begin{cases} 6c + 14h = 86.178 & \text{hexane's molecular weight} \\ 8c + 18h = 114.232 & \text{octane's molecular weight} \end{cases}$$

b. To eliminate c when you add the equations, you must make its coefficients additive inverses, that is, numbers with opposite signs. If you multiply the hexane equation by 4 and the octane equation by -3, then you get two new equations set up for elimination.

$$\begin{array}{lll} 4(6c + 14h) = 4(86.178) \rightarrow & 24c + 56h = \quad 344.712 & \text{Multiply both sides by 4.} \\ -3(8c + 18h) = 3(114.232) \rightarrow & \underline{-24c - 54h = -342.696} & \text{Multiply both sides by } -2. \\ & \quad 2h = 2.016 & \text{Add the equations.} \\ & \quad h = 1.008 & \text{Divide both sides by 2 and reduce.} \end{array}$$

To find the value of c, you could substitute 1.008 for h in one of your original equations and solve for c, as you did in the previous lesson. Or you could go back to the original equations and use elimination on h. If you multiply the hexane equation by -9 and the octane equation by 7, then you get two equations set up to eliminate h.

SHARING IDEAS

Have groups share different methods for eliminating C in Step 5, and present their ideas for Step 9.

Ask if there's another way to check the solutions. Students might suggest using a ruler to measure the paper clips and pennies.

Assessing Progress
Watch for the ability to follow directions, make careful measurements, and set up a system of two linear equations.

EXAMPLE B

This example shows how to eliminate a variable in a system of equations by multiplying the equations by different numbers.

You may want to review the chemical terms *mole* and *molecular weight.* 1 mole of a substance means 6.02×10^{23} molecules of that substance (just as 1 dozen eggs means 12 eggs). The molecular weight of a substance is the weight in grams of 1 mole of that substance.

[Ask] "How do you choose the numbers by which to multiply both sides of the equation?" [Students might use the coefficients themselves, the opposites of coefficients, or the products or least common multiples of coefficients.]

Part b of the solution illustrates how an arrow can mean *implies.* Remind students that the implies arrow is more than a vague link between steps in a process. If required, again stress the need for logical correctness in checking a solution.

You might ask students to solve the same system by substitution and think about when one method is preferable to the other. A few more examples might help them see that substitution is preferable if the coefficient of one variable in one equation is 1.

Closing the Lesson

As needed, point out that the **elimination method** is a fourth method of solving a system of equations, along with substitution, graphing, and tables. The last two often give only approximations, but elimination and substitution give exact solutions.

BUILDING UNDERSTANDING

Students practice the elimination method for solving systems of equations and deepen their understanding of the graphical meaning of a solution to a system.

ASSIGNING HOMEWORK

Essential	1–4, 7, 9
Performance assessment	4, 11, 13
Portfolio	9, 14
Journal	4, 12
Group	5, 10, 12
Review	15–18

Helping with the Exercises

Exercise 1 Students may be surprised to see that when they multiply both sides of an equation by the same number the graph stays the same.

$$-9(6c + 14h) = -9(86.178) \rightarrow -54c - 126h = -775.602$$ Multiply both sides by -9.

$$7(8c + 18h) = 7(114.232) \rightarrow \underline{56c + 126h = 799.624}$$ Multiply both sides by 7.

$$2c = 24.022$$ Add the equations.

$$c = 12.011$$ Divide both sides by 2 and reduce.

c. The solution to the system is (12.011, 1.008). So the atomic weight of carbon is 12.011 grams per mole and the atomic weight of hydrogen is 1.008 grams per mole. Check your answers by substituting them into the original equations.

$$\begin{aligned} 8c + 18h &= 114.232 \\ 8(12.011) + 18(1.008) &\stackrel{?}{=} 114.232 \\ 96.088 + 18.144 &\stackrel{?}{=} 114.232 \\ 114.232 &= 114.232 \end{aligned}$$

$$\begin{aligned} 6c + 14h &= 86.178 \\ 6(12.011) + 14(1.008) &\stackrel{?}{=} 86.178 \\ 72.066 + 14.112 &\stackrel{?}{=} 86.178 \\ 86.178 &= 86.178 \end{aligned}$$

Because you get true statements for both equations, the solution checks.

There is no single right order to the steps in solving a system of equations, so you can start by choosing a variable that's easy to eliminate. You can use both elimination and substitution if that's easiest. Always check your solution by substituting into the original system.

EXERCISES

You will need your graphing calculator for Exercises **9** and **12**.

Practice Your Skills

1. Consider the equation $5x + 2y = 10$.

a. Solve the equation for y and sketch the graph. ⓐ

b. Multiply the equation $5x + 2y = 10$ by 3, and then solve for y. How does the graph of this equation compare with the graph of the original equation? Explain your answer. ⓐ

2. Use the equation $5x - 2y = 10$ to find the missing coordinate of each point.

a. $(6, a)$ ⓗ (6, 10) **b.** $(-4, b)$ ⓐ $(-4, -15)$ **c.** $(c, 25)$ (12, 25) **d.** $(d, -5)$ $(0, -5)$

3. Solve each system of equations by elimination. Show your work.

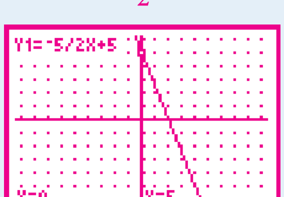
a. $\begin{cases} 6x + 5y = -20 \\ -6x - 10y = 25 \end{cases}$

You can simply add the equations as they are to eliminate the x-terms: $-5y = 5$, $y = -1$; $6x = -15$, $x = -2.5$. The solution is $(-2.5, -1)$.

b. $\begin{cases} 5x - 4y = 23 \\ 7x + 8y = 5 \end{cases}$

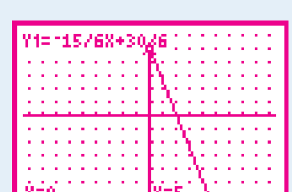
You can multiply the first equation by 2 to eliminate the y-terms: $17x = 51$, $x = 3$; $8y = -16$, $y = -2$. The solution is $(3, -2)$.

1a. $y = \dfrac{10 - 5x}{2}$, or $y = 5 - \dfrac{5x}{2}$



$[-9.4, 9.4, 1, -6.2, 6.2, 1]$

1b. $y = \dfrac{30 - 15x}{6}$, or $y = 5 - \dfrac{5x}{2}$



The graph is the same as the graph for 1a. Both equations are equivalent to $y = 5 - \frac{5}{2}x$.

4. Anisha turned in this quiz in her algebra class.

a. What method did she use? substitution

b. What is missing from her solution?

c. Complete Anisha's solution. $y = -1$; $(4, -1)$

> Anisha ______ Score ______
>
> Solve this system:
> $y = x - 5$
> $3y + 2x = 5$
>
> Solution:
> $3(x - 5) + 2x = 5$
> $3x - 15 + 2x = 5$
> $5x = 20$
> $x = 4$

5. Consider this system of equations:

$$\begin{cases} 3x + 7y = -8 \\ 5x + 8y = -6 \end{cases}$$

In 5a and b, tell how you can eliminate each variable when you combine the equations by addition.

a. the x-term @

b. the y-term

Reason and Apply

6. List the different ways you have learned to solve the system. Then choose one method and find the solution. @ The solution is $(2, -2)$. You can

$$\begin{cases} 3x + 7y = -8 \\ 5x + 8y = -6 \end{cases}$$

(1) solve for y and graph, then look for the point where the lines intersect;
(2) solve for y, create tables, and zoom in to where the y-values are equal;
(3) solve one equation for y (or x) and substitute into the other; or
(4) multiply the equations and add them to eliminate x or y.

7. Solve each system using the elimination method.

a. $\begin{cases} 2x + y = 10 \\ 5x - y = 18 \end{cases}$ $(4, 2)$

b. $\begin{cases} 3x + 5y = 4 \\ 3x + 7y = 2 \end{cases}$ $(3, -1)$

c. $\begin{cases} 2x + 9y = -15 \\ 5x + 9y = -24 \end{cases}$ $(-3, -1)$

8. In 8a–c, solve each equation for y and sketch a graph of the result on the same set of axes.

a. $x - 2y = 6$ @ $y = -3 + 0.5x$

b. $3x + 4y = 8$ @ $y = 2 - 0.75x$

c. Graph the equation you get from adding the original two equations in 8a and b. @ $y = 7 - 2x$

d. What does the graph tell you? @ The solution of the system is also a solution of the sum of the equations.

9. Refer to this system from Example A to answer each question.

$$\begin{cases} x + y = 163 \\ x - y = 33 \end{cases}$$

a. Solve each equation for y and enter these new equations into your calculator. Use the window [0, 150, 10, 0, 150, 10] to graph this system. $y = 163 - x$ and $y = -33 + x$

b. Use the elimination method to find the y-value of the solution. Enter the resulting equation into Y_3 and add it to your graph from 9a. @

c. Use elimination to find the x-value of the solution. Draw a vertical line on the graph to represent the equation you found in 9b.

d. Describe what you notice about the four lines on your screen and explain why this happens. The four lines intersect at the same point, (98, 65); the solution to the system must satisfy all the equations—the original equations in the system and any new equations created by combining pairs of equations.

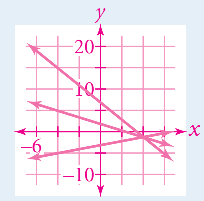
Exercise 4 Students might say that a check is part of what's missing. Encourage this kind of thinking.

4b. The y-value is missing from her solution.

5a. Multiply the first equation by -5 and the second equation by 3, or multiply the first equation by 5 and the second equation by -3.

5b. Multiply the first equation by -8 and the second equation by 7, or multiply the first equation by 8 and the second equation by -7.

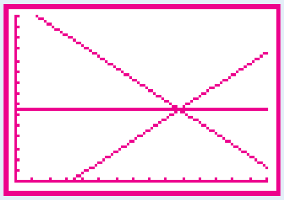
8.



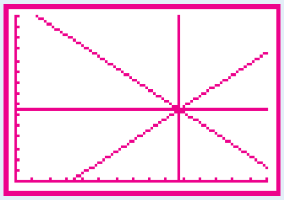
Exercise 9 Students continue to explore the connections between the elimination method and the graphs representing the equations, and they discover that all the equations intersect in a single point.

In 9c, students are asked to draw a vertical line on their calculators. If they don't remember how to do this, refer them to Calculator Note 4B.

9b. $2y = 130$, $y = 65$



9c. $2x = 196$, $x = 98$



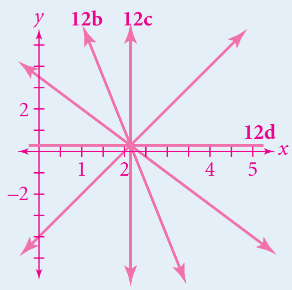
Exercise 10 If students are stuck, you might want to suggest that they write the second equation in the form $4x + ay = b$. For any value of a, there's a value of b that gives an equation satisfied by $(5, 2)$.

Ask students how to tell graphically that there are infinitely many solutions to the second equation. [Infinitely many lines pass through the given solution point.] Try to get them talking and writing about their ideas. Ask if only the line with slope 4 is a possibility. If all the students think so, suggest that, given that the coefficient of y can be any number, there's not enough information to determine the slope. Welcome challenges to both ideas, because the discussion can bring out misconceptions and help deepen understanding of how slope is calculated.

Exercise 11 Allow a variety of approaches to solving this system. You might use the opportunity to review the notion of equivalent equations.

Exercise 12 As needed, **[Ask]** "Why is it important that all these lines pass through that point of intersection?" [The fact that the graph of any sum or constant multiple of these equations passes through this point shows why the elimination method gives a solution.]

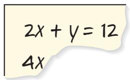
12a. $y = 3.5 - 1.5x$ and $y = -4 + 2x$



12b. $y = 11 - 5x$; this line passes through the point where the two original equations intersect.

12c. $x = \frac{15}{7}$; this line passes through the point where the two original equations intersect.

10. Part of Adam's homework paper is missing. If $(5, 2)$ is the only solution to the system shown, write a possible equation that completes the system. ⓗ



Answers will vary. Substitute $(5, 2)$ for x and y in $4x + ay = b$ to get $20 + 2a = b$. One possibility is $4x - 3y = 14$.

11. Consider this system of equations:

$$\begin{cases} 2x - 5y = 12 \\ 6x - 15y = 36 \end{cases}$$

a. By what number can you multiply which equation to eliminate the x-term when you combine the equations by addition? Do this multiplication. Multiply the first equation by -3; $-6x + 15y = -36$.

b. What is the sum of these equations? $0 = 0$

c. What is the solution to the system? There are infinitely many solutions.

d. How can you predict this result by examining the original equations? One equation is a multiple of the other.

12. ***Mini-Investigation*** Consider the system

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$$\begin{cases} 3x + 2y = 7 \\ 2x - y = 4 \end{cases}$$

a. Solve each equation for y and graph the result on your calculator. Sketch the graph on your paper.

b. Add the two original equations and solve the resulting equation for y. Add this graph to your graph from 12a. What do you notice?

c. Multiply the second original equation by 2, then add this to the first equation. Solve this equation for x and add its graph to your graph from 12a. What do you notice?

d. Multiply the first original equation by 2 and the second by -3, then add the results. Solve this equation for y and add its graph to 12a. What do you notice?

e. What is the solution to the system of equations? How does this point relate to the graphs you drew in 12a–d? $\left(\frac{15}{7}, \frac{2}{7}\right)$; this is the intersection point of all the lines in 12a–d.

f. Write a few sentences summarizing any conjectures you can make based on this exercise.

13. APPLICATION The school's photographer took pictures of couples at this year's prom. She charged \$3.25 for wallet-size pictures and \$10.50 for portrait-size pictures.

a. Write a system of equations representing the fact that Crystal and Dan bought a total of 10 pictures for \$61.50. ⓐ

b. Solve this system and explain what your answer means. They bought six wallet-size pictures and four portrait-size.

14. APPLICATION Automobile companies advertise two rates for fuel mileage. City mileage is the rate of fuel consumption for driving in stop-and-go traffic. Highway mileage is the rate for driving at higher speeds for long periods of time.

Cynthia's new car gets 17 mi/gal in the city and 25 mi/gal on the highway. She drove 220 miles on 11 gallons of gas.

a. Define variables and write a system of equations for the gallons burned at each mileage rate. ⓐ

12d. $y = \frac{2}{7}$; this line passes through the point where the two original equations intersect.

12f. Answers will vary. If two equations intersect in a point, any combination of multiples of the two equations intersects in the same point. That's why the elimination method works.

13a. $\begin{cases} w + p = 10 \\ 3.25w + 10.50p = 61.50 \end{cases}$

14a. Let c represent gallons burned in the city and h represent gallons burned on the highway.

$$\begin{cases} c + h = 11 \\ 17c + 25h = 220 \end{cases}$$

b. Solve this system and explain the meaning of the solution. @ (6.875, 4.125); 6.875 gal in the city, 4.125 gal on the highway

c. Find the number of city miles and highway miles Cynthia drove. @

d. Check your answers. @

Review

0.1 **15.** For each pair of fractions, name a fraction that lies between them. Answers will vary. Samples are

a. $\frac{1}{2}$ and $\frac{3}{4}$ $\frac{5}{8}$ **b.** $\frac{2}{3}$ and $\frac{7}{8}$ $\frac{3}{4}$ **c.** $-\frac{1}{4}$ and $-\frac{1}{5}$ $-\frac{9}{40}$ **d.** $\frac{7}{11}$ and $\frac{5}{6}$ $\frac{2}{3}$

e. Describe a strategy for naming a fraction between any two fractions.

3.5 **16.** APPLICATION When you go up a mountain, the temperature drops about 4 degrees Fahrenheit for every 1000 feet you ascend.

a. While climbing a trail on Mt. McKinley in Alaska, Marsha intended to record the elevation and temperature at three locations. Complete the table for her.

Marsha's Climb

	Elevation (ft)	Temperature (°F)
Start	4,300	78
Rest station	7,800	64
Highest point	11,900	47.6

b. Write an equation to model the relationship between elevation and temperature. Explain the meanings of the slope and y-intercept.

c. Mt. McKinley is 20,320 feet tall. On the day Marsha was climbing, how cold was it at the summit?
At the summit the temperature was 13.9°F.

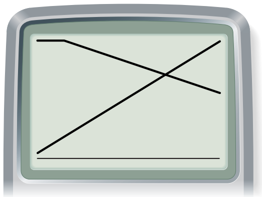
This mountain climber is ascending Mt. McKinley in Denali National Park, Alaska.

4.3 **17.** Write an equation in point-slope form using the given information.

a. A line that passes through the point $(5, -3)$ and has slope -2. $y = -3 - 2(x - 5)$

b. A line that passes through the point $(-3, 7)$ and has slope 2.5. $y = 7 + 2.5(x + 3)$

5.1 **18.** The graph at right pictures distances from a motion sensor for two walkers. (Walker A starts at 0.5 ft and walks at 1 ft/s. Walker B waits at 10.5 ft until 1 second has passed and then walks at 0.5 ft/s.)



a. Write an equation for each walk. (*Hint:* Walker B's distance can be recorded in two segments. The first is $y = 10.5$ when $x \leq 1$.)

b. When and where do they meet? They meet 7.5 ft from the sensor, when 7 s have passed.

c. When is Walker B farther from the sensor than Walker A?
Walker B is farther from the sensor than Walker A for all times up to, but not including, 7 s.

14c. $\frac{17 \text{ mi}}{\text{gal}} \cdot 6.875 \text{ gal} \approx$ 117 city mi, $\frac{25 \text{ mi}}{\text{gal}} \cdot 4.125 \text{ gal} \approx$ 103 hwy mi

14d. check:
$$\begin{cases} 6.875 + 4.125 = 11 \\ 17(6.875) + 25(4.125) = 220 \end{cases}$$
and $117 + 103 = 220$

15e. Sample answer: Find a common denominator, select a new numerator between the other two, and reduce.

16b. $T = 95.2 - 0.004E$; the slope is the rate of change in temperature for each increase of 1 ft in elevation, and the y-intercept (in this case, T-intercept) is the temperature that day at sea level in the same area.

18a. Walker A: $y = 0.5 + x$; Walker B: $y = 10.5$ when $x \leq 1$ and $y = 10.5 - 0.5(x - 1)$, or $y = 11 - 0.5x$, when $x > 1$

LESSON
5.4

PLANNING

LESSON OUTLINE

First day:

- 20 min Introduction, Example A
- 30 min Investigation

Second day:

- 10 min Investigation
- 5 min Sharing
- 10 min Example B
- 5 min Closing
- 20 min Exercises

MATERIALS

- Row Operations in a Matrix (T), *optional*
- Calculator Note 5A

TEACHING

Other methods for solving systems of equations employ matrices. Variations on the Gaussian elimination method are used by computers handling many linear equations in many variables.

This lesson is optional. Omit it unless you taught Lesson 1.8 on matrices. Some of the exercises can be solved using the elimination method if you want to give your students more practice with that method.

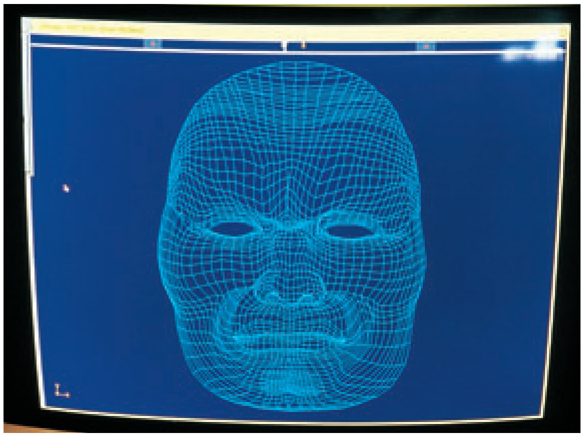
Note that the text uses *numerals* instead of *numbers* to characterize coefficients and constants in the original equations that are transferred to the matrix. Students need to realize that x and y are numbers or sets of numbers.

LESSON
5.4 Solving Systems of Equations Using Matrices

The essence of mathematics is not to make simple things complicated but to make complicated things simple.

STANLEY GUDDER

In Lesson 1.8, you learned how to enter, display, and use matrices to organize and analyze data. In this lesson you will use matrices to solve systems of equations. This method of solving systems of equations is similar to the elimination method, but using matrices may be quicker because you can keep track of equations using a shorter notation. Computers and graphing calculators can solve complex systems of equations entered in matrix form.



Software that renders 3-D computer-generated images uses matrices to organize data. This program graphs thousands of points and lines to draw the contours of a person's face.

If you look only at the numerals in a system of equations in standard form $ax + by = c$—that is, the coefficients of both variables and the constant terms—you have a matrix with two rows and three columns. If you have a system with both equations in standard form $ax + by = c$, you can write a matrix for the system:

$$\begin{cases} 5x + 3y = -1 \\ 2x - 6y = 50 \end{cases} \qquad \begin{bmatrix} 5 & 3 & -1 \\ 2 & -6 & 50 \end{bmatrix}$$

The numerals in the first equation match the numerals in the first row, and the numerals in the second equation match the numerals in the second row. But what does the solution look like in a matrix? The solution to the system above is $(4, -7)$, or $x = 4$ and $y = -7$. You want the rows of the solution matrix to represent the equations. So you can rewrite each equation to get the numerals for each row of the solution matrix:

$$\begin{matrix} x = 4 \\ y = -7 \end{matrix} \quad \rightarrow \quad \begin{matrix} x + 0y = 4 \\ 0x + y = -7 \end{matrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -7 \end{bmatrix}$$

LESSON OBJECTIVES

- Represent a linear system with a matrix
- Use the method of Gaussian elimination for solving systems of linear equations
- Use the calculator to solve systems of linear equations

NCTM STANDARDS

	CONTENT		PROCESS
	Number	✓	Problem Solving
✓	Algebra	✓	Reasoning
	Geometry		Communication
	Measurement		Connections
	Data/Probability	✓	Representation

In the elimination method, you combined equations and multiplied them by numbers. In much the same way, you can modify the rows of a matrix by performing **row operations** on each number in those rows.

Row Operations in a Matrix

- Multiply (or divide) all numbers in a row by a nonzero number.
- Add all numbers in a row to corresponding numbers in another row.
- Add a multiple of the numbers in one row to the corresponding numbers in another row.
- Exchange two rows.

You can do these operations on the rows of a matrix to change the starting matrix into a solution matrix. The goal is to get a diagonal of 1's in the matrix with 0's above and below, like this:

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$$

The ordered pair (a, b) is the solution, if one exists, to the system.

EXAMPLE A

Solve this system of equations using matrices:

$$\begin{cases} x - 2y = 3 \\ 3x + y = 23 \end{cases}$$

▶ Solution

Copy the numerals from each equation into each row of the matrix. Then use row operations to transform it into the solution matrix.

$$\begin{cases} x - 2y = 3 \\ 3x + y = 23 \end{cases} \rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 3 & 1 & 23 \end{bmatrix}$$

Add −3 times row 1 to row 2.

−3 times row 1	→	−3	6	−9
+ row 2	→	3	1	23
New row 2	→	0	7	14

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 7 & 14 \end{bmatrix}$$

Divide row 2 by 7.

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Add 2 times row 2 to row 1.

2 times row 2	→	0	2	4
+ row 1	→	1	−2	3
New row 1	→	1	0	7

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \end{bmatrix}$$

One Step

Show students how to represent a system like $2x + y = 11$ and $6x - 5y = 9$ as a matrix of coefficients. Then challenge them to change the numbers in the matrix to indicate a solution to the system derived by elimination. As groups finish, suggest that they try the same approach on a system that has infinitely many or no solutions such as $y = 3x + 4$ and $2y - 6x = 8$ or $y = 2x + 1$ and $y - 2x = 3$. During Sharing, lead students to formalize the row operations and the diagonalization procedure.

INTRODUCTION

As needed, point out how the row operations mimic the operations on equations used in the previous lesson.

Implies Arrows

One bad habit some algebra students can begin to develop is writing a long string of equalities, such as

$3x + 5 = 8 = 3x = 3 = x = 1$

Among other inaccuracies, this statement claims that $8 = 1$. The equal signs have two different meanings. Some show equations, but the ones between equations mean *implies*. If indeed the solution is to be written on one line, have students write the word *implies* or use its abbreviation, an arrow:

$3x + 5 = 8 \Rightarrow 3x = 3 \Rightarrow x = 1$

Stress that implies arrows must be used carefully in mathematical statements. When the student text uses a regular arrow, it often means "next you do this." The first of the arrows used near the bottom of page 296 could be written as an implies arrow, but the second one should not be.

EXAMPLE A

This example illustrates how a system can be solved using row operations on a matrix.

Guiding the Investigation

[Language] A diagonalized matrix has the form

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$$

Step 2 **[Alert]** Students often want to subtract 6 from all entries in row 2. Ask them to write out the corresponding equations to see that they've represented subtracting $6x$ and $6y$ from one side but 6 from the other side.

Step 2 Add -3 times row 1 to row 2 and record the sum in row 2 to get $\begin{bmatrix} 2 & 1 & 11 \\ 0 & -8 & -24 \end{bmatrix}$.

Step 3 **[Alert]** Some students may try to add 9 to all entries in row 2.

Step 3 Divide row 2 by -8 to get $\begin{bmatrix} 2 & 1 & 11 \\ 0 & 1 & 3 \end{bmatrix}$.

Step 4 Add -1 times row 2 to row 1 and write the answer in row 1:

$$\begin{bmatrix} 2 & 1 & 11 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 8 \\ 0 & 1 & 3 \end{bmatrix}$$

Step 5 Divide row 1 by 2 to get $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \end{bmatrix}$; it means $x = 4$ and $y = 3$, so the solution is (4, 3).

SHARING IDEAS

Point out the quotation opening the lesson. Ask in what sense using matrices is simplifying.

Have a group share their solution from Step 6. Ask the class to compare it to the matrix method.

Assessing Progress

You can assess students' familiarity with matrix-related terms such as row, column, and entry and their understanding of the elimination method and how to check a solution to a system of equations. You also will be able to tell how deeply they understand which systems have no solution and which have infinitely many solutions.

Use the solution matrix to write the equations:

$$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{array}{lll} 1x + 0y = 7 & \text{or} & x = 7 \\ 0x + 1y = 2 & \text{or} & y = 2 \end{array}$$

The solution to the system is (7, 2).

Investigation
Diagonalization

In this investigation you will see how to combine row operations in your solution process.

Consider the system of equations

$$\begin{cases} 2x + \ \ y = 11 \\ 6x - 5y = \ \ 9 \end{cases}$$

Step 1 $\begin{bmatrix} 2 & 1 & 11 \\ 6 & -5 & 9 \end{bmatrix}$; the numerals from the first equation; the numerals from the second equation

Step 6 Solution will mirror Steps 2–5; multiply the first equation by -3 and add it to the second equation to get $-8y = -24$, or $y = 3$; substitute $y = 3$ into the first equation and solve for x to get $x = 4$.

Step 1 Write the matrix for this system. What does the first row contain? The second row?

Step 2 Describe how to use row operations to get 0 as the first entry in the second row. Write this matrix.

Step 3 Next, get 1 as the second number in the second row of your matrix from Step 2.

Step 4 Use row operations on the matrix from Step 3 to get 0 as the second number in row 1.

Step 5 Next, get 1 as the first number of row 1 of your matrix from Step 4. Tell what this matrix means, and give the solution to the system.

Step 6 Check your solution using elimination. Eliminate x first.

Step 7 Look at the first three rules for Row Operations in a Matrix. How do they correspond to steps in the elimination process? In the elimination process, you can multiply an equation by a constant without affecting the solution. You can also add two equations together, ideally to eliminate a variable, without affecting the solution. And, you can also multiply one or both equations by a constant, then add them together.

History CONNECTION

German mathematician Carl Friedrich Gauss (1777–1855) made many contributions to mathematics, including developing the elementary row operations on matrices. In his honor the process of solving systems with matrices is sometimes called "Gaussian elimination." To learn more about Gauss, see www.keymath.com/DA .

Matrices are useful for solving systems involving large numbers. Here is another example.

EXAMPLE B

On Friday, 3247 people attended the county fair. The entrance fee for an adult was \$5, and for a child 12 or under the fee was \$3. The fair collected a total of \$14,273. How many of the total attendees were adults and how many were children?

▶ Solution

Use A for the number of adults attending the fair and C for the number of children attending. Use these variables to write a system of equations and solve it using matrices. The attendance is the number of adults and children at the fair. So the first equation is $A + C = 3247$. The fair collected $5A$ dollars for A adults and $3C$ dollars for C children in attendance. The total collected is $5A + 3C$, so the second equation is $5A + 3C = 14273$.

With one equation describing attendance at the fair, and another describing ticket money collected, the system is

$$\begin{cases} A + C = 3247 \\ 5A + 3C = 14{,}273 \end{cases} \longrightarrow \begin{bmatrix} 1 & 1 & 3247 \\ 5 & 3 & 14{,}273 \end{bmatrix}$$

Use row operations to find the solution.

$-5R_1 + R_2$

Add -5 times row 1 to row 2 to get new row 2. $\begin{bmatrix} 1 & 1 & 3247 \\ 0 & -2 & -1962 \end{bmatrix}$

$R_2/-2$

Divide row 2 by -2. $\begin{bmatrix} 1 & 1 & 3247 \\ 0 & 1 & 981 \end{bmatrix}$

$-1R_2 + R_1$

Add -1 times row 2 to row 1 to get new row 1. $\begin{bmatrix} 1 & 0 & 2266 \\ 0 & 1 & 981 \end{bmatrix}$

The final matrix shows that $A = 2266$ and $C = 981$. So there were 2266 adults and 981 children at the fair on Friday.

To check this solution, substitute 2266 for A and 981 for C into the original equations.

$$\begin{aligned} A + C &= 3247 \\ 2266 + 981 &\stackrel{?}{=} 3247 \\ 3247 &= 3247 \end{aligned} \qquad \begin{aligned} 5A + 3C &= 14{,}273 \\ 5(2266) + 3(981) &\stackrel{?}{=} 14{,}273 \\ 11{,}330 + 2943 &\stackrel{?}{=} 14{,}273 \\ 14{,}273 &= 14{,}273 \end{aligned}$$

These are true statements, so the solution checks.

With row operations on matrices, you now have five methods to solve systems of linear equations. Like elimination and substitution, row operations on matrices give exact solutions. With practice, you will develop a sense of when it is easiest to use each solution method. The form of the equation often makes some methods easier to use than others. If an equation is solved for y, then it is easiest to use the substitution method. If the equations are in standard form, then it is probably easiest to solve by elimination or by using matrices.

EXAMPLE B

This example provides another illustration of solving a system of equations using row operations. It resembles an old puzzle, so some students might suggest a non-algebraic solution. (Each of the 3247 people paid \$3, bringing in \$9741. The rest of the revenue, \$4532, came from the adults, each of whom paid an additional \$2. So there must have been 2266 adults.) Praise good thinking like this. Challenge students to represent that reasoning with row operations on a matrix. They'll get a sequence like that in the example, except that they'll eliminate the 3 in the second row.

Notice that in each solution, A and C are verified by showing that both sides of the equation equal the same thing.

Closing the Lesson

We now have seen five ways of solving systems of equations. Two of them—graphing and tables—often give only approximations. For exact answers, use substitution, elimination, or matrices.

BUILDING UNDERSTANDING

Students work with matrix representations of systems of equations and use row operations to solve those systems.

ASSIGNING HOMEWORK

Essential	1–6, 9, 10
Performance assessment	9, 10
Portfolio	9–11
Journal	7
Group	8, 11, 15
Review	12–14

Helping with the Exercises

1a. $\begin{cases} 2x + 1.5y = 12.75 \\ -3x + 4y = 9 \end{cases}$

4. Divide row 1 by 4.2: $\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 5.25 \end{bmatrix}$.

Multiply row 2 by -1: $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -5.25 \end{bmatrix}$.

Solution: $(3, -5.25)$.

Exercise 6 If students have difficulty keeping track of this calculation, encourage them to write out intermediate steps.

8 times row 1	24	16	231.2
-3 times row 2	-24	-15	-223.8
Sum (new row 2)	0	1	7.4

EXERCISES

You will need your graphing calculator for Exercise **8.**

Practice Your Skills

1. Write a system of equations whose matrix is

a. $\begin{bmatrix} 2 & 1.5 & 12.75 \\ -3 & 4 & 9 \end{bmatrix}$ @

b. $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 2 & 0 \end{bmatrix}$ $\begin{cases} \frac{1}{2}x = \frac{1}{2} \\ -x + 2y = 0 \end{cases}$

c. $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 0 \end{bmatrix}$ $\begin{cases} 2x + 3y = 1 \\ 2y = 0 \end{cases}$

2. Write the matrix for each system.

a. $\begin{cases} x + 4y = 3 \\ -x + 2y = 9 \end{cases}$ @ $\begin{bmatrix} 1 & 4 & 3 \\ -1 & 2 & 9 \end{bmatrix}$

b. $\begin{cases} 7x - y = 3 \\ 0.1x - 2.1y = 3 \end{cases}$ $\begin{bmatrix} 7 & -1 & 3 \\ 0.1 & -2.1 & 3 \end{bmatrix}$

c. $\begin{cases} x + y = 3 \\ x + y = 6 \end{cases}$ $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 6 \end{bmatrix}$

3. Write each solution matrix as an ordered pair.

a. $\begin{bmatrix} 1 & 0 & 8.5 \\ 0 & 1 & 2.8 \end{bmatrix}$ @ $(8.5, 2.8)$

b. $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{13}{16} \end{bmatrix}$ $\left(\frac{1}{2}, \frac{13}{16}\right)$

c. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ $(0, 0)$

4. Use row operations to transform $\begin{bmatrix} 4.2 & 0 & 12.6 \\ 0 & -1 & 5.25 \end{bmatrix}$ into the form $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$.

Write the solution as an ordered pair. ⓗ

5. Consider the system

$$\begin{cases} y = 7 - 3x \\ y = 11 - 2(x - 5) \end{cases}$$

a. Convert each equation to the standard form $ax + by = c$. @ $\begin{cases} 3x + y = 7 \\ 2x + y = 21 \end{cases}$

b. Write a matrix for the system. @ $\begin{bmatrix} 3 & 1 & 7 \\ 2 & 1 & 21 \end{bmatrix}$

Reason and Apply

6. Give the missing description, matrix, and equations for each step of the process below. Give the solution as an ordered pair.

Description	Matrix	System equations
The matrix for $\begin{cases} 3x + 2y = 28.9 \\ 8x + 5y = 74.6 \end{cases}$	$\begin{bmatrix} 3 & 2 & 28.9 \\ 8 & 5 & 74.6 \end{bmatrix}$	$3x + 2y = 28.9$ $8x + 5y = 74.6$
Add 8 times row 1 to -3 times row 2 and put the result in row 2.	$\begin{bmatrix} 3 & 2 & 28.9 \\ 0 & 1 & 7.4 \end{bmatrix}$	$3x + 2y = 28.9$ $y = 7.4$
Add -2 times row 2 to row 1 and put the result in row 1.	$\begin{bmatrix} 3 & 0 & 14.1 \\ 0 & 1 & 7.4 \end{bmatrix}$	$3x = 14.1$ $y = 7.4$
Divide row 1 by 3. The solution is $(4.7, 7.4)$.	$\begin{bmatrix} 1 & 0 & 4.7 \\ 0 & 1 & 7.4 \end{bmatrix}$	$x = 4.7$ $y = 7.4$

7. **APPLICATION** Each day, Sal prepares a large basket of self-serve tortilla chips in his restaurant. On Monday, 40 adult patrons and 15 child patrons ate 10.8 kg of chips. On Tuesday, 35 adult patrons and 22 child patrons ate 12.29 kg of chips. Sal wants to know whether adults or children eat more chips on average.

a. Organize the information into a table. @

b. Define variables and write a system of equations. @

c. Write a matrix for the system.

d. Solve the system by transforming the matrix into the solution matrix $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \end{bmatrix}$.

e. Write a sentence that describes the real-world meaning of the solution to the system. **Each adult ate an average of about 0.15 kg (150 g) of chips, and each child ate an average of 0.32 kg (320 g) of chips.**

8. Your graphing calculator probably has built-in row operations to transform a matrix into its solution form. Transform this matrix using row operations on your calculator.

[▶ See Calculator Note 5A. ◀]

$$\begin{bmatrix} 8 & 7 & -1 \\ 3 & -1 & -4 \end{bmatrix}$$

9. **APPLICATION** Zoe must ship 532 tubas and 284 kettledrums from her warehouse to a store across the country. A truck rental company offers two sizes of trucks. A small truck will hold 5 tubas and 7 kettledrums. A large truck will hold 12 tubas and 4 kettledrums. If she wants to fill each truck so that the cargo won't shift, how many small and large trucks should she rent?

a. Define variables and write a system of equations to find the number of small trucks and the number of large trucks Zoe needs to ship the instruments. (*Hint:* Write one equation for each instrument.) ⓗ

b. Write a matrix that represents the system. @

c. Perform row operations to transform the matrix into a solution matrix.

d. Write a sentence describing the real-world meaning of the solution. **Zoe should order 20 small trucks and 36 large trucks.**

10. **APPLICATION** Will is baking a new kind of bread. He has two different kinds of flour. Flour X is enriched with 0.12 mg of calcium per gram; Flour Y is enriched with 0.04 mg of calcium per gram. Each loaf has 300 g of flour, and Will wants each loaf to have 30 mg of calcium. How much of each type of flour should he use for each loaf?

a. Will wrote this system of equations:

$$\begin{cases} x + y = 300 \\ 0.12x + 0.04y = 30 \end{cases}$$

Give a real-world meaning to the variables x and y, and describe the meaning of each equation.

10a. x **represents the number of grams of Flour X used in each loaf, and** y **represents the number of grams of Flour Y used in each loaf. The first equation sums the amount of each type of flour to get the total amount of flour in the loaf, and the second equation sums the amount of calcium contributed by each type of flour to get the total amount of calcium.**

b. Write a matrix for the system. **10b.** $\begin{bmatrix} 1 & 1 & 300 \\ 0.12 & 0.04 & 30 \end{bmatrix}$ **10c.** $\begin{bmatrix} 1 & 0 & 225 \\ 0 & 1 & 75 \end{bmatrix}$

c. Find the solution matrix.

d. Explain the real-world meaning of the solution. **Will should mix 225 g of Flour X with 75 g of Flour Y.**

Exercise 7 Systems of equations show the power of algebra, which was developed to make arithmetical reasoning easier. Algebra is not needed for this problem:

> Each day Sal prepares a large basket of self-serve tortilla chips in his restaurant. One day, his 45 patrons ate 10.8 kg of chips. Sal knows from experience that each child eats 0.2 kg of chips and each adult eats 0.3 kg. How many adults and how many children patronized his restaurant that day?

In this case, you can think of dividing the 10.8 kg into portions of 0.2 kg each. There will be 54 of these portions. There were only 45 patrons, though, so 1.8 kg went to adults. Each adult eats 0.1 kg more than a child, so the 1.8 kg was distributed among 18 adults. The rest of the 45 patrons (27) must have been children.

However, for Exercise 7, Sal knows that on Monday 40 adults and 15 children ate a total of 10.8 kg of chips, and on Tuesday 35 adults and 22 children ate 12.29 kg of chips. Without using algebra, the thinking needed to solve this problem would be quite difficult.

Exercise 8 You may want to skip the calculator method if your students are using TI-73 or TI-82 calculators.

8.

```
[A]
      [[1 0 -1]
       [0 1  1 ]]
```

Exercise 9 If students have difficulty deciding what the equations should look like, ask what they want to find (to help them determine the variables) and what they know (to set up the equations).

9a–c. See page 302.

7a.

	Adults	Children	Total (kg)
Monday	40	15	10.8
Tuesday	35	22	12.29

7b. Let x represent the average weight of chips an adult eats and y represent the average weight of chips a child eats. The system is

$$\begin{cases} 40x + 15y = 10.8 \\ 35x + 22y = 12.29 \end{cases}.$$

7c. $\begin{bmatrix} 40 & 15 & 10.8 \\ 35 & 22 & 12.29 \end{bmatrix}$

7d. Add -35 times row 1 to 40 times row 2 and put the result in row 2: $\begin{bmatrix} 40 & 15 & 10.8 \\ 0 & 355 & 113.6 \end{bmatrix}$.

Divide row 2 by 355: $\begin{bmatrix} 40 & 15 & 10.8 \\ 0 & 1 & 0.32 \end{bmatrix}$.

Add -15 times row 2 to row 1: $\begin{bmatrix} 40 & 0 & 6 \\ 0 & 1 & 0.32 \end{bmatrix}$.

Divide row 1 by 40: $\begin{bmatrix} 1 & 0 & 0.15 \\ 0 & 1 & 0.32 \end{bmatrix}$.

11. **APPLICATION** On Monday a group of students started on a three-day bicycle tour covering a total of 286 km. On Tuesday they cycled 7 km less than on Monday. On Wednesday they traveled 24 km less than on Tuesday.

a. Write a system of three linear equations representing this trip. Use m, t, and w to represent the distances in kilometers they cycled on Monday, Tuesday, and Wednesday, respectively. Write each equation in the form $am + bt + cw = d$. @

b. Write a 3×4 matrix to model this system of equations. Describe what the rows and columns of your matrix represent. @

c. List and describe a sequence of matrix row operations that will produce a matrix of the form The sequence of row operations will vary; the solution matrix is

$$\begin{bmatrix} 1 & 0 & 0 & ? \\ 0 & 1 & 0 & ? \\ 0 & 0 & 1 & ? \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 108 \\ 0 & 1 & 0 & 101 \\ 0 & 0 & 1 & 77 \end{bmatrix}.$$

d. What is the solution to this problem?
They cycled 108 km on Monday, 101 km on Tuesday, and 77 km on Wednesday.

Review

1.8 **12.** **APPLICATION** These matrices show the cost, in dollars, of a 1-day ticket and a 3-day ticket for an adult, a teen, and a child at two amusement parks, Tivoli and Hill.

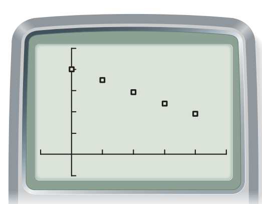
$$\begin{array}{c} \textbf{1-day ticket} \\ \begin{array}{l} \\ \text{Adult} \\ \text{Teen} \\ \text{Child} \end{array} \begin{array}{c} \begin{array}{cc} \text{Tivoli} & \text{Hill} \end{array} \\ \begin{bmatrix} 31 & 28 \\ 26 & 24 \\ 21 & 16 \end{bmatrix} \end{array} \end{array} \qquad \begin{array}{c} \textbf{3-day ticket} \\ \begin{array}{l} \\ \text{Adult} \\ \text{Teen} \\ \text{Child} \end{array} \begin{array}{c} \begin{array}{cc} \text{Tivoli} & \text{Hill} \end{array} \\ \begin{bmatrix} 72 & 65 \\ 55 & 55 \\ 45 & 35 \end{bmatrix} \end{array} \end{array}$$

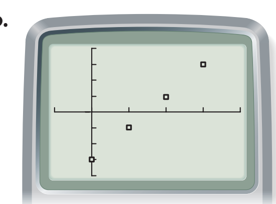
a. Write a matrix equation displaying the difference in cost between a 3-day ticket and a 1-day ticket.

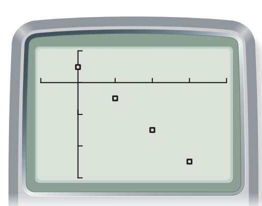
b. Which type of ticket is the better deal and why?

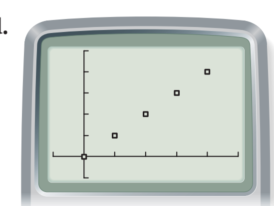
c. Which type of ticket should you buy if you are in the park for only 2 days?

3.2 **13.** Write a recursive sequence for the y-coordinates of the points shown on each graph. On each graph one tick mark represents one unit. ⓗ

a.  4, Ans − 0.5

b.  −3, Ans + 2

c.  1/2, Ans − 1

d.  0, Ans + 1

Exercise 11 This problem asks students to write a system of equations using three variables. Be sure students realize they need to use zeros as placeholders for missing variables when they write a system in matrix form.

11a. $\begin{cases} m + t + w = 286 \\ m - t \qquad = 7 \\ \qquad t - w = 24 \end{cases}$

11b. $\begin{bmatrix} 1 & 1 & 1 & 286 \\ 1 & -1 & 0 & 7 \\ 0 & 1 & -1 & 24 \end{bmatrix}$

The rows represent each equation. The columns represent the coefficients of each variable and the constants.

12a. $\begin{bmatrix} 72 & 65 \\ 55 & 55 \\ 45 & 35 \end{bmatrix} - \begin{bmatrix} 31 & 28 \\ 26 & 24 \\ 21 & 16 \end{bmatrix} = \begin{bmatrix} 41 & 37 \\ 29 & 31 \\ 24 & 19 \end{bmatrix}$

12b. If you are planning to be in the park for 3 days, then the 3-day ticket is a much better deal. If you bought three 1-day tickets, the cost would be $\begin{bmatrix} 93 & 84 \\ 78 & 72 \\ 21 & 48 \end{bmatrix}$.

12c. If you are going to be in the park for 2 days, the cost of two 1-day tickets would be $\begin{bmatrix} 62 & 56 \\ 52 & 48 \\ 42 & 32 \end{bmatrix}$. This is less than the cost of the 3-day ticket, so if you are going for only 2 days, you should buy two 1-day tickets.

9a. Let x represent the number of small trucks and y represent the number of large trucks. The system is $\begin{cases} 5x + 12y = 532 \\ 7x + 4y = 284 \end{cases}$.

9b. $\begin{bmatrix} 5 & 12 & 532 \\ 7 & 4 & 284 \end{bmatrix}$

9c. Solution steps will vary; $\begin{bmatrix} 5 & 12 & 532 \\ 7 & 4 & 284 \end{bmatrix} \rightarrow \begin{bmatrix} -16 & 0 & -320 \\ 0 & -64 & -2304 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 20 \\ 0 & 1 & 36 \end{bmatrix}$.

4.4 **14.** At the Coffee Stop, you can buy a mug for \$25 and then pay only \$0.75 per hot drink.

a. What is the slope of the equation that models the total cost of refills? What is the real-world meaning of the slope? **Slope: 0.75; the slope is the cost per drink once you've bought the mug.**

b. Use the point (33, 49.75) to write an equation in point-slope form that models this situation. $y = 49.75 + 0.75(x - 33)$

c. Rewrite your equation in intercept form. What is the real-world meaning of the y-intercept? $y = 25 + 0.75x$**; the y-intercept is the cost of buying the mug.**

15. Over 2000 years ago the Chinese developed column equation matrices as a method to solve linear equations. The numerals of each equation are arranged in columns instead of rows. Then you use the biancheng (translated as "multiply throughout") and zhichu (translated as "direct reduction") rules of operation to solve the system.

Rules of Operation

1. Biancheng: Multiply the numerals of the left column by the numeral at the top of the right column.
2. Zhichu: Subtract the right column from the resulting left column repeatedly until you get a 0 at the top.

For example, represent this system as a column equation matrix.

$$\begin{cases} 2x + y = 11 \\ 6x - 5y = 9 \end{cases} \rightarrow \begin{bmatrix} 2 & 6 \\ 1 & -5 \\ 11 & 9 \end{bmatrix}$$

Biancheng: Multiply the first column by 6 (highest top row numeral).

$$\begin{aligned} 2(6) &\rightarrow 12 \\ 1(6) &\rightarrow 6 \\ 11(6) &\rightarrow 66 \end{aligned}$$

Zhichu: Subtract the right column from the left column twice.

$$\begin{array}{ccccccc} 12 & - & 6 & - & 6 & \rightarrow & 0 \\ 6 & - & (-5) & - & (-5) & \rightarrow & 16 \\ 66 & - & 9 & - & 9 & \rightarrow & 48 \end{array}$$

Write a new equation and solve for y.

$$\begin{aligned} 16y &= 48 \\ y &= 3 \end{aligned}$$

Substitute and solve for x.

$$\begin{aligned} 2x + 3 &= 11 \\ x &= 4 \end{aligned}$$

Now use a Chinese column equation matrix to solve the system

$$\begin{cases} x - 2y = 3 \\ 3x + y = 23 \end{cases}$$ @

$$\begin{bmatrix} 1 & 3 \\ -2 & 1 \\ 3 & 23 \end{bmatrix} \rightarrow \begin{array}{cc} 3 & -3 \\ -6 & -1 \\ 9 & -23 \end{array} \rightarrow \begin{array}{c} 0 \\ -7 \\ -14 \end{array}$$

$-7y = -14, y = 2; x = 7$

(Jean-Claude Martzloff, *A History of Chinese Mathematics,* 1997, pp. 252–254; Lĭ Yăn and Dù Shírán, *Chinese Mathematics, a Concise History,* 1987, pp. 46–48)

Exercise 15 This historical problem shows another way to solve a system of equations. Assign this exercise as enrichment for students who enjoy a challenge. It may confuse students who are just learning the regular row operations.

LESSON

5.5

PLANNING

LESSON OUTLINE

One day:

5 min	Introduction
20 min	Investigation
5 min	Sharing
10 min	Examples
10 min	Exercises

MATERIALS

- rope marked with whole numbers from −10 to 10 about 1 or 2 ft apart, or paper or chalk number line with two markers, *optional* (one per group)
- Toe the Line (W, T), *optional*
- Number Line (W), *optional*

TEACHING

Inequalities help model problems for situations described by words like *greater than, less than, no more than, at least,* and so on. Students gain understanding kinesthetically of how multiplying or dividing by a negative number reverses the direction of the inequality. Otherwise, solving inequalities is much like solving equations by balancing.

INTRODUCTION

You might ask the class for other phrases modeled by inequalities, such as *no more than, above, over,* and *more than.* Point out that the larger value is at the larger side of the inequality symbol.

LESSON

5.5

Inequalities in One Variable

Some material may be inappropriate for children under 13.

DESCRIPTION OF PG-13 RATING, MOTION PICTURE ASSOCIATION OF AMERICA

Drink at least six glasses of water a day. Store milk at temperatures below 40°F. Eat snacks with fewer than 20 calories. Spend at most $10 for a gift. These are a few examples of inequalities in everyday life. In this lesson you will analyze situations involving inequalities in one variable and learn how to find and graph their solutions.

An **inequality** is a statement that one quantity is less than or greater than another. You write inequalities using these symbols:

less than	$<$	less than or equal to	$\leq$
greater than	$>$	greater than or equal to	$\geq$

Sometimes you need to translate everyday language into the phrases you see in the table above. Here are some examples.

Everyday phrase	Translation	Inequality
at least six glasses	The number of glasses is greater than or equal to 6.	$g \geq 6$
below 40°	The temperature is less than 40°.	$t < 40$
fewer than 20 calories	The number of calories is less than 20.	$c < 20$
at most $10	The price of the gift is less than or equal to $10.	$p \leq 10$
between 35° and 120°	35° is less than the temperature and the temperature is less than 120°.	$35 < t < 120$

History CONNECTION

Thomas Harriot (1560–1621) introduced the symbols of inequality $<$ and $>$. Pierre Bouguer (1698–1758) first used the symbols $\leq$ and $\geq$ about a century later.

(Florian Cajori, *A History of Mathematics,* 1985)

You solve inequalities very much like you solve equations. You use the same strategies—adding or subtracting the same quantity to both sides, multiplying both sides by the same number or expression, and so on. However, there is one exception you need to remember when solving inequalities. You will explore this exception in the investigation.

LESSON OBJECTIVES

- Write and solve one-variable inequalities and interpret the results based on real-world situations
- Graph solutions to one-variable inequalities on a number line, showing whether they are strict inequalities
- Interpret an interval graphed on a number line as an inequality sentence
- Learn the sign-change rule for multiplying or dividing both sides of a one-variable inequality by a negative number

NCTM STANDARDS

	CONTENT		PROCESS
✓	Number		Problem Solving
✓	Algebra	✓	Reasoning
✓	Geometry	✓	Communication
	Measurement	✓	Connections
	Data/Probability	✓	Representation

Investigation
Toe the Line

You will need

- chalk or a tape measure to mark a segment

In this investigation you will analyze properties of inequalities and discover some interesting results.

First you'll act out operations on a number line.

Procedure Note

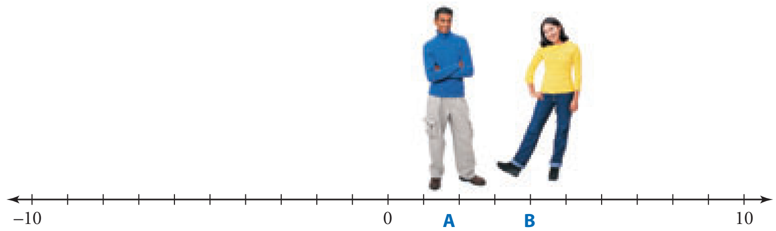
The announcer calls out operations for Walkers A and B. The walkers perform operations on their numbers by walking to the resulting values on the number line. The recorder logs the position of each walker after each operation.

Step 1 In your group, choose an announcer, a recorder, and two walkers. The two walkers make a number line on the floor with marks from −10 to 10. The announcer and recorder make a table with these column headings and twelve rows. The operations to use as row headings are Starting number, Add 2, Subtract 3, Add −2, Subtract −4, Multiply by 2, Subtract 7, Multiply by −3, Add 5, Divide by −4, Subtract 2, and Multiply by −1.

Operation	Walker A's position	Inequality symbol	Walker B's position
Starting number	2		4
Add 2			

keymath.com/DA

Step 2 Read the Procedure Note. As a trial, act out the first operation in the table: Walker A simply stands at 2 on the number line, and Walker B stands at 4.



Enter the inequality symbol into the table that describes the relative position of Walkers A and B on the number line. Be sure you have written a true inequality.

Step 3 Call out the operations. After the walkers calculate their new numbers, record the operation and walkers' positions in the next row.

Step 4 As a group, discuss which inequality symbol to enter into each cell of the third column.

Next you'll analyze what each operation does to the inequality.

Step 5 What happens to the walkers' relative positions on the number line when the operation adds or subtracts a positive number? A negative number? Does anything happen to the direction of the inequality symbol?

Step 6 What happens to the walkers' relative positions on the number line when the operation multiplies or divides by a positive number? Does anything happen to the inequality symbol? **The walkers' positions stretch from side to side, but the walkers do not switch relative positions; the inequality doesn't change.**

Guiding the Investigation

One Step
Ask walkers to follow the directions in Step 1. Each group should write inequalities to describe the result of each step and then look for patterns. Encourage them to note when the direction of inequality changes.

Step 1 In a group consisting of only three members, one can serve as both recorder and announcer. If time is a problem, use ropes prepared ahead of time. If space is also a problem, use the Number Line worksheet and move two markers instead of two people.

Step 2 You may want to discuss the term *relative position*. It means the position of the walkers in relation to each other. The one on the right is greater. Relative position does not indicate how far apart the walkers are.

Step 4 Correct students who use the term *equation* to refer to an inequality.

Step 5 The walkers' positions shift right and left but maintain the same distance apart; the inequality symbol doesn't change.

Steps 1–4

Operation	A's position	Inequality symbol	B's position
Start	2	<	4
Add 2	4	<	6
Subtract 3	1	<	3
Add −2	−1	<	1
Subtract −4	3	<	5
Multiply by 2	6	<	10

Operation	A's position	Inequality symbol	B's position
Subtract 7	−1	<	3
Multiply by −3	3	>	−9
Add 5	8	>	−4
Divide by −4	−2	<	1
Subtract 2	−4	<	−1
Multiply by −1	4	>	1

Step 7 Students may recall that the change in direction of the inequality occurred when the two walkers passed each other.

Step 8 As needed, note that multiplying by -1 reflects a point across zero. You may want students to circle rows in the table where the inequality switches directions.

Step 9 Encourage students to continue using decimals and other fractions. If time permits, explore the effects of operations such as square roots and powers.

SHARING IDEAS

Note the opening quotation, and ask how it relates to the lesson. To bring out the strictness of the inequality, ask if 13-year-olds are included in the advice.

Ask students what patterns they saw in Step 9. Watch for confusion about the relative position of two negative numbers—that is, the one closer to 0 is larger although it has the smaller magnitude.

Inequalities with $<$ or $>$ are *strict* inequalities. Entertain the common but infrequently articulated question of why you'd ever need any other kind of inequality. After all, if two numbers are equal, you can use an equal sign. Students may get ideas about this question from the table in the introduction.

[Ask] "How might inequalities be graphed?" Encourage all participation, not just "right answers." Through repeatedly asking what makes one graphing method better than another, you may be able to get enough ideas from students to make Example A unnecessary.

Assessing Progress
Watch for comfort with negative integers and operations on them, especially subtraction of negatives. Assess familiarity with the number line, the ability to follow systematic instructions, and the willingness to look for patterns in data.

Step 7 The walkers switch relative positions; the inequality symbol reverses direction.

Step 8 Multiplying or dividing by a negative number reverses the direction of the inequality. All elementary operations—add, subtract, multiply, and divide—with positive numbers maintain the direction of the inequality. Adding or subtracting negative numbers also preserves the inequality. Multiplying by a number greater than 1 or dividing by a fraction between -1 and 1 increases the distance between walkers.

Step 7 What happens to the walkers' relative positions on the number line when the operation multiplies or divides by a negative number? Does the inequality symbol change directions?

Step 8 Which operations on an inequality reverse the inequality symbol? Does it make any difference which numbers you use? Consider fractions and decimals as well as integers.

Step 9 Check your findings about the effects of adding, subtracting, multiplying, and dividing by the same number on both sides of an inequality by creating your own table of operations and walkers' positions.

In square dancing, a caller tells the dancers which steps to take. Their maneuvers depend on their relative positions.

This example will show you how to graph solutions to inequalities.

EXAMPLE A

Graph each inequality on a number line.

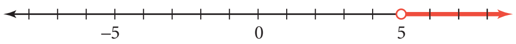
a. $t > 5$

b. $x \leq -1$

c. $-2 \leq x < 4$

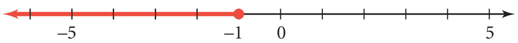
▸ Solution

a. Any number greater than 5 satisfies the inequality $t > 5$. So $5.0001 > 5$, $7\frac{1}{2} > 5$, and $1{,}000{,}000 > 5$ are all true statements. You show this by drawing an arrow through the values that are greater than 5.



The open circle at 5 excludes 5 from the solutions because $5 > 5$ is not a true statement.

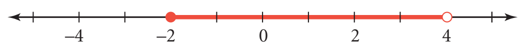
b. The inequality $x \leq -1$ reads, "x is less than or equal to -1." The solid circle at -1 includes the value -1 in the solutions because $-1 \leq -1$ is a true statement.



EXAMPLE A

In this example, students see how to graph the solution of an inequality on a number line. Remind students that an open circle is used to represent the endpoint of a strict inequality and that a filled circle is used to represent the endpoint of an inequality that includes equality. Be sure students understand that the ray indicates an unbounded set of numbers.

c. This statement is a **compound inequality.** It says that -2 is less than or equal to x and that x is less than 4. So the graph includes all values that are greater than or equal to -2 but less than 4. The solid circle at -2 includes -2 in the solutions because $-2 \leq -2$ is true. The open circle at 4 excludes 4 from the solutions because $4 < 4$ is not true.



When you graph inequalities, always label 0 on the number line as a point of reference.

EXAMPLE B

Erin says, "I lose 15 minutes of sleep every time the dog barks. Last night I got less than 5 hours of sleep. I usually sleep 8 hours." Find the number of times Erin woke up.

To solve the problem, let x represent the number of times Erin woke up, and write an inequality.

Solve the inequality and graph your solutions.

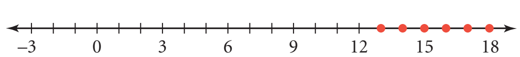
▶ Solution

The number of hours Erin slept is 8 hours, minus $\frac{1}{4}$ hour times x, the number of times she woke up. The total is less than 5 hours. So the inequality is $8 - 0.25x < 5$.

Solve the inequality for x. Remember to reverse the inequality symbol if you multiply or divide by a negative number.

$8 - 0.25x < 5$	Original inequality.
$8 - 0.25x - 8 < 5 - 8$	Subtract 8 from both sides of the inequality.
$-0.25x < -3$	Evaluate.
$\frac{-0.25x}{-0.25} > \frac{-3}{-0.25}$	Divide both sides by -0.25, and reverse the inequality symbol.
$x > 12$	Divide.

The dog woke her up more than 12 times. However, Erin can only wake up a whole number of times, so the solution might be more accurately written as, "$x > 15$, where x is a whole number." The solution graph of this statement looks like this:



Is there a maximum number of times that Erin can be woken up during the night? You'll explore this question in Exercise 15.

EXAMPLE B

This example goes through the complete modeling process: representing a real-life problem with an inequality, solving the inequality, and interpreting the solution in the original context.

Writing the inequality may be the hardest step. Be sure students see the expression $8 - 0.25x$ as the amount of sleep Erin got. **[Ask]** "Why does the inequality include $0.25x$ instead of $15x$?" [The units of time need to be the same for every number in the expression.] As students read through the solution, be sure they notice that the direction of the inequality sign reverses.

Once students have seen the solution to the inequality, complete the modeling process by asking what solution it gives to the original problem.

Closing the Lesson

Inequalities help model problems for situations described by phrases such as *greater than, less than, no more than,* and *at least.* Solving inequalities is much like solving equations, except that multiplying or dividing by a negative number reverses the direction of the inequality. The solutions to an inequality can be represented by a ray, illustrating an unbounded set of numbers in the solution. If an open circle marks the ray's endpoint, the endpoint is not included in the solution. A solid circle indicates inclusion of the endpoint.

BUILDING UNDERSTANDING

Students practice setting up and solving inequalities and graphing their solutions.

ASSIGNING HOMEWORK

Essential	1–4, 6, 7, 11, 12
Performance assessment	8, 9
Portfolio	8
Journal	12
Group	10, 12, 13, 14
Review	5, 16–18

Helping with the Exercises

Exercise 2 You might ask students to make up real-world problems that are modeled by the given inequalities.

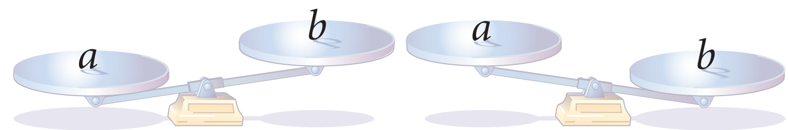
2a. Answers will vary, but the values must be > 8.

2d. Values must be $< \frac{120}{13}$ or $9\frac{3}{13}$ (≈ 9.2308).

Exercise 3 3d and 3e introduce students to compound inequalities. As needed, explain to students that, because the x is written between two inequality symbols, x is greater than some number *and* x is less than some other number.

Exercise 4 **[ELL]** Some students may not know how the phrases *more than, no more than,* and *not greater than* correspond to the phrases *greater than* and *less than or equal to* used earlier in the lesson.

Working with inequalities is very much like working with equations. An equation shows a balance between two quantities, but an inequality shows an imbalance. The important thing to remember is that multiplying and dividing both sides of an equation by a negative number tips the scales in the opposite direction.



$a > b$ $\qquad$ $a < b$

EXERCISES

You will need your graphing calculator for Exercise **16.**

Practice Your Skills

1. Tell what operation on the first inequality gives the second one, and give the answer using the correct inequality symbol.

a. $3 < 7$
$4 \cdot 3 \;\square\; 7 \cdot 4$ @ Multiply by 4; $12 < 28$.

b. $5 \le 12$
$-3 \cdot 5 \;\square\; 12 \cdot -3$ Multiply by -3; $-15 \ge -36$.

c. $-4 \ge x$
$-4 + (-10) \;\square\; x + (-10)$ @ Add -10; $-14 \ge x - 10$.

d. $b + 3 > 15$
$b + 3 - 8 \;\square\; 15 - 8$ Subtract 8; $b - 5 > 7$.

e. $24d < 32$
$\frac{24d}{3} \;\square\; \frac{32}{3}$ @ Divide by 3; $8d < 10\frac{2}{3}$.

f. $24x \le 32$
$\frac{24x}{-3} \;\square\; \frac{32}{-3}$ Divide by -3; $-8x \ge -10\frac{2}{3}$.

2. Find three values of the variable that satisfy each inequality.

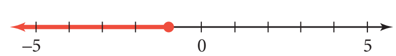
a. $5 + 2a > 21$ @

b. $7 - 3b < 28$ Values must be > -7.

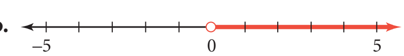
c. $-11.6 + 2.5c < 8.2$ Values must be < 7.92.

d. $4.7 - 3.25d > -25.3$

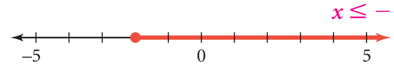
3. Give the inequality graphed on each number line.

a. 

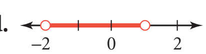
@ $x \le -1$

b. 

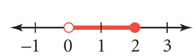
$x > 0$

c. 

$x \ge -2$

d. 

@ $-2 < x < 1$

e. 

$0 < x \le 2$

4. Translate each phrase into symbols.

a. 3 is more than x $\quad 3 > x$

b. y is at least -2 @ $\quad y \ge -2$

c. z is no more than 12 $\quad z \le 12$

d. n is not greater than 7 $\quad n \le 7$

5. Solve each equation for y.

a. $3x + 4y = 5.2$ $\quad y = \frac{5.2 - 3x}{4} = 1.3 - 0.75x$

b. $3(y - 5) = 2x$ $\quad y = \frac{2x}{3} + 5$, or $\frac{2x + 15}{3}$

Reason and Apply

6. Solve each inequality and show your work.

a. $4.1 + 3.2x > 18$ ⓐ $x > 4.34375$, or $\frac{139}{32}$

b. $7.2 - 2.1b < 4.4$ $b > 1.\overline{3}$

c. $7 - 2(x - 3) \geq 25$ $x \leq -6$

d. $11.5 + 4.5(x + 1.8) \leq x$ $x \leq -5.6$

7. Solve each inequality and graph the solutions on a number line.

a. $3x - 2 \leq 7$

b. $4 - x > 6$ ⓐ

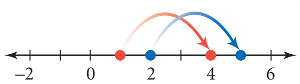
c. $3 + 2x \geq -3$

d. $10 \leq 2(5 - 3x)$

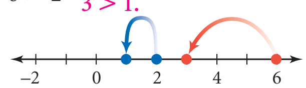
8. Ezra received \$50 from his grandparents for his birthday. He makes \$7.50 each week for odd jobs he does around the neighborhood. Since his birthday, he has saved more than enough to buy the \$120 gift he wants to buy for his parents' 20th wedding anniversary. How many weeks ago was his birthday? ⓗ

9. For each graph, tell what operation moves the two points in the inequality to their new positions. Write the new inequality, stating the position of the red dot first.

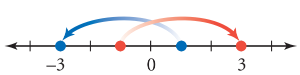
a. $1 < 2$ ⓐ Add 3 to both sides; $4 < 5$.



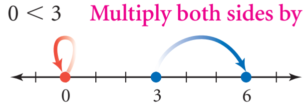
b. $6 > 2$ Divide both sides by 2 (or multiply by 0.5); $3 > 1$.



c. $-1 < 1$ Multiply both sides by -3; $3 > -3$.



d. $0 < 3$ Multiply both sides by 2; $0 < 6$.



10. Tell whether each inequality is true or false for the given value.

a. $x - 14 < 9, x = 5$ $-9 < 9$ is true.

b. $3x \geq 51, x = 7$ $21 \geq 51$ is false.

c. $2x - 3 < 7, x = 5$ $7 < 7$ is false.

d. $4(x - 6) \geq 18, x = 12$ $24 \geq 18$ is true.

11. Solve each inequality. Explain the meaning of the result. On a number line, graph the values of x that make the original inequality true.

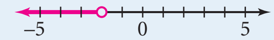
a. $2x - 3 > 5x - 3x + 3$ ⓐ

b. $-2.2(5x + 3) \geq -11x - 15$

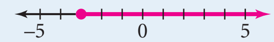
12. Data collected by a motion sensor will vary slightly in accuracy. A given sensor has a known accuracy of ± 2 mm (0.002 m), and a distance is measured as 2.834 m. State this distance and accuracy as an inequality statement.
$2.834 - 0.002 \leq x \leq 2.834 + 0.002$; $2.832 \leq x \leq 2.836$ m

7a. $x \leq 3$

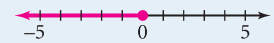
7b. $x < -2$



7c. $x \geq -3$



7d. $x \leq 0$



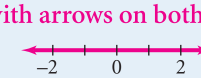
Exercise 8 A student who's very comfortable with dimensional analysis might go directly to the inequality $7.5w > 70$. Be encouraging.

8. $50 + 7.5w > 120$; $w > 9.\overline{3}$; Ezra has been saving for at least 10 wk.

Exercise 9 Here's another way to visualize operations on inequalities. You may want to model a problem like this one to be sure students understand that the directions of the arrows indicate movement of the two numbers relative to the number line.

11a. The variable x drops out of the inequality, leaving $-3 > 3$, which is never true. So the original inequality is not true for any number x. The graph would be an empty number line, with no points filled in.

11b. The variable x drops out of the inequality, leaving $-6.6 \geq -15$, which is always true. So the original inequality is true for any number x. The graph would be a line with arrows on both ends.



Exercise 12 This exercise reviews accuracy, introduced in Lesson 1.7. Students may need help writing the compound inequality.

Exercise 13 **[ELL]** Again, be sure all students know the meaning of the phrases that describe the symbols.

13a. $d \leq 30$ (d for dollars spent on CDs)

13b. $h \geq 48$ (h for height of riders)

13. You read the inequality symbols, $<$, $\leq$, $>$, and $\geq$, as "is less than," "is less than or equal to," "is greater than," and "is greater than or equal to," respectively. But you describe everyday situations with different expressions. Identify the variable in each statement and give the inequality to describe each situation. ⓗ

a. I'll spend no more than \$30 on CDs this month.

b. You must be at least 48 inches tall to go on this ride.

c. Three or more people make a carpool. $p \geq 3$ (p for people in carpool)

d. No one under age 17 will be admitted without a parent or guardian. $a \geq 17$ (a for age of person admitted)

14. The table gives equations that model the three vehicles' distances in the Investigation On the Road Again from Lesson 3.2. The variable x represents the time in minutes since all three vehicles began traveling, and y represents the distance in miles from Flint.

Equation	Vehicle
$y = 220 - 1.2x$	minivan
$y = 35 + 0.8x$	sports car
$y = 1.1x$	pickup truck

a. What question is represented by the inequality statement $35 + 0.8x \geq 131$? When is the sports car 131 or more miles away from Flint?

b. What is the solution to the inequality $35 + 0.8x \geq 131$? $x \geq 120$

c. What question is represented by the statement $220 - 1.2x < 35 + 0.8x$?

d. What is the solution to the inequality $220 - 1.2x < 35 + 0.8x$? $x > 92.5$

14c. When is the minivan closer than the sports car to Flint?

15. In Example B, the inequality $8 - 0.25x < 5$ was written to represent the situation where Erin slept less than 5 hours, and her sleep time was 8 hours minus 0.25 hour for each time the dog barked. However, Erin can't sleep less than 0 hours, so a more accurate statement would be the compound inequality $0 \leq 8 - 0.25x < 5$. You can solve a compound inequality in the same way you've solved other inequalities; you just need to make sure you do the same operation to all *three* parts. Solve this inequality for x and graph the solution.

15. $12 < x \leq 32$

Review

2.7 **16.** List the order in which you would perform these operations to get the correct answer.

a. $72 - 12 \cdot 3.2 = 33.6$ Multiply 12 by 3.2 to get 38.4. Subtract 38.4 from 72 to get 33.6.

b. $2 + 1.5\left(3 - 5^2\right) = -31$

c. $21 \div 7 - 6 \div 2 = 0$

Divide 21 by 7 to get 3 and divide 6 by 2 to get 3. Subtract 3 from 3 to get 0.

16b. Square 5 to get 25. Subtract 25 from 3 to get -22. Multiply -22 by 1.5 to get -33. Add -33 to 2 to get -31.

17a. 0.37 ENTER

Ans + 0.23 ENTER, ENTER, . . .

Weight (oz)	Rate (\$)
1	0.37
2	0.60
3	0.83
4	1.06
5	1.29
6	1.52
7	1.75
8	1.98
9	2.21
10	2.44
11	2.67

3.2 **17.** The table shows the 2004 U.S. postal rates for letters, large envelopes, and small packages.

U.S. Postal Rates

Weight	Rate
First ounce or fraction of an ounce	\$0.37
Each additional ounce or fraction	\$0.23

(U.S. Postal Service, *www.usps.com*)

a. Use a recursive routine to create a table that shows the cost of sending letters weighing from 0 to 11 ounces. @

b. Use 1-ounce units on the horizontal axis to plot the postal costs. @

c. Kasey has drawn a line through the points on her graph. What real-world meaning does this line have? Is a line useful in this situation? Why or why not? @

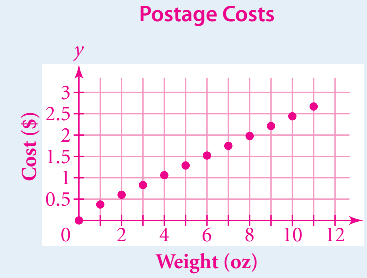
d. What is the cost of sending a 10.5-ounce parcel? @ \$2.67

4.4 **18.** Use the distributive property to rewrite each expression without using parentheses.

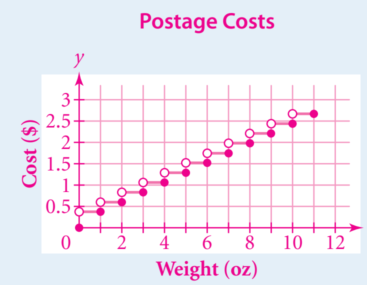
a. $-2(x + 8)$ $-2x - 16$ **b.** $4(0.75 - y)$ $3 - 4y$

c. $-(z - 5)$ $-z + 5$

17b.



17c. A line would mean that the cost would pass through each amount between the different increments. For example, if a package weighed 0.5 oz, you would pay \$0.185. However, the cost increases discretely. To show this, draw segments for each integral ounce. Note the open and closed circles.



project

TEMPERATURES

Temperatures for your city vary depending on the time of day, season, and its location. Weather reports give the daily high and low temperatures and often compare them with the record temperatures in the past 100 years.

Research the range of temperatures for your geographic area. What are the record highs and lows? What are the record temperatures for a specific day, say, your birthday? How do the altitude and location of your area affect these temperatures?

Compare your results to temperatures on the moon. Research the temperatures of other planets such as Venus, Mars, and Pluto. What factors affect these data sets? Are the temperatures given in degrees Fahrenheit or degrees Celsius? Be sure to convert all data to the same units before comparing. Describe your findings with inequalities and graphs in a paper or give a presentation.

This view from the *Apollo II* spacecraft shows Earth above the lunar terrain.

Your project should include

- Your hometown high and low temperatures.
- Algebraic expressions with compound inequalities.
- Clearly labeled graphs.

Some people think it may be possible to live on another planet or moon someday. Based on your findings, what do you think?

Supporting the project

MOTIVATION

How do record temperatures in your geographic area compare with those on other planets or on the moon?

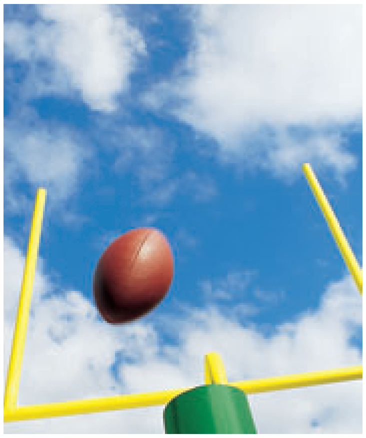
OUTCOMES

- Record highs and lows for the area overall, as well as temperatures for a specific day, are all given in the same units.
- The report comments on how the altitude and geographic location affect the temperatures.
- Temperatures for the moon or for other planets are given and compared.
- Factors influencing extraterrestrial temperatures are given.
- The paper or presentation includes clearly labeled graphs as well as inequalities, including compound inequalities.
- Claims about the possibility of living on another planet or on the moon are consistent with the data given.

LESSON 5.6

Graphing Inequalities in Two Variables

In Lesson 5.5, you learned to graph inequalities in one variable on a number line. However, some situations, such as the number of points a football team scores by touchdowns and field goals, require more than one variable. In this lesson you will learn to graph inequalities in two variables on a coordinate plane.

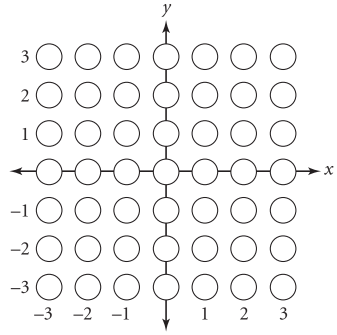


You have graphed equations like $y = 1 + 0.5x$. In the following investigation you will learn how to graph inequalities such as $y < 1 + 0.5x$ and $y > 1 + 0.5x$.

Investigation
Graphing Inequalities

You will need

- the worksheet Graphing Inequalities Grids



First you'll make a graph from one of four statements.

i. $y \square 1 + 0.5x$ **ii.** $y \square -1 - 2x$

iii. $y \square 1 - 0.5x$ **iv.** $y \square 1 - 2x$

Step 1 Each member of the group should choose a different statement from above.

Step 2 Evaluate the right side of your statement for $x = -3$. For each circle in the first column on the graph, fill in $>$ if the y-value of the point is greater than your value, $=$ if the values are equal, and $<$ if the y-value is less than your value. The first column for each graph should be filled in as shown in the answer for Step 3.

LESSON 5.6

PLANNING

LESSON OUTLINE

One day:

20 min	Investigation
5 min	Sharing
10 min	Examples
5 min	Closing
10 min	Exercises

MATERIALS

- Graphing Inequalities Grids (W)
- Graphing Inequalities (T), *optional*
- Calculator Notes 5B, 5C
- Sketchpad demonstration Graphing Inequalities, *optional*

TEACHING

Solutions to inequalities involving two variables can be visualized as points in half-planes. If you have The Geometer's Sketchpad, you might introduce this lesson with the Sketchpad demonstration Graphing Inequalities.

Guiding the Investigation

One Step

Ask students to divide statements i, ii, iii, and iv among group members. For each one, fill in each circle of the grid worksheet with the symbol $<$, $>$, or $=$ that should go into the box to make a true statement when that point's coordinates are substituted for x and y. Ask students to look for patterns. As you circulate, encourage students to check points with fractional coordinates, to redraw the graphs without the circles, and to use dotted lines as boundaries of strict inequalities.

LESSON OBJECTIVES

- Solve two-variable inequalities for y
- Graph inequalities on the coordinate plane and show the solutions as the intersection of two half-planes
- Interpret graphs of half-planes and write corresponding inequalities

NCTM STANDARDS

	CONTENT		PROCESS
	Number	✓	Problem Solving
✓	Algebra	✓	Reasoning
	Geometry	✓	Communication
	Measurement	✓	Connections
✓	Data/Probability	✓	Representation

Step 3 | Repeat Step 2 for $x = -2, -1, 0, 1, 2,$ and 3.

Next you'll analyze the results of your graph.

Step 4 | What do you notice about the circles filled with the equal sign? Describe any other patterns you see.

Step 5 | Test a point with fractional or decimal coordinates that is not represented by a circle on the grid. Compare your result with the symbols on the same side of the line of equal signs as your point.

Step 6 | Draw a set of xy-axes, with scales from -3 to 3 on each axis. Under the graph, write your statement with the "less than" symbol, $<$. Shade the region of points that makes your statement true. If the points on the line make an inequality true, draw a solid line through them. If not, draw a dashed line. Repeat this step for each of the remaining symbols ($>, \leq, \geq, =$).

Step 4 The circles filled with equal signs form a line. The "greater than" symbols, $>$, are all above the line, and the "less than" symbols, $<$, are all below the line.

Step 5 Coordinates will vary. The symbol will be the same as the symbols on the same side of the line of equal signs.

Finally, you'll draw general conclusions by comparing graphs in your group.

Step 7 | Compare your graphs with those of others in your group. What graphs require a solid line? A dashed line?

Step 8 | What graphs require shading? Shading above the line? Below the line?

Step 9 | Discuss how to use one point to check the graph of an inequality.

Step 7 The graphs for the symbols $=, \leq,$ and $\geq$ require a solid line because points on the line satisfy the relationship; the strict inequalities, $<$ and $>$, require a dashed line.

The graph of the solutions to a single inequality is called a **half-plane** because it includes all the points in the coordinate plane that fall on one side of the boundary line.

EXAMPLE A

Graph the inequality $2x - 3y > 3$, and check to see whether each point is part of the solution.

i. $(3, -2)$
ii. $(3, 1)$
iii. $(-1, 2)$
iv. $(-2, -3)$

▶ Solution

To graph the inequality, first solve it for y:

$2x - 3y > 3$	Original inequality.
$-3y > 3 - 2x$	Subtract $2x$ from both sides.
$y < -1 + \frac{2}{3}x$	Divide both sides by -3 and reverse the inequality symbol.

Step 7 **[Ask]** "Are broken lines and solid lines related to open circles and filled circles?" [Broken lines, like open circles, represent strict inequalities. Solid lines, like filled circles, include equality.]

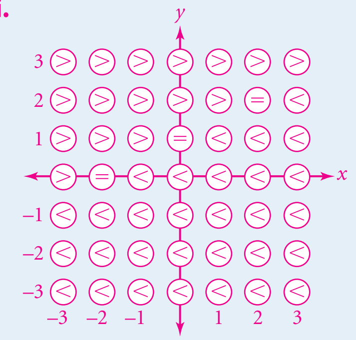
Step 8 The graphs for the symbols $<, >, \leq,$ and $\geq$ require shading. For $>$ and $\geq$, shade above the line. Shade below the line for $<$ and $\leq$.

Step 9 Answers will vary. Substitute into the inequality the coordinates from one point on one side of the line, say $(0, 0)$. If the point satisfies the inequality, shade that side of the line. If not, shade the other side.

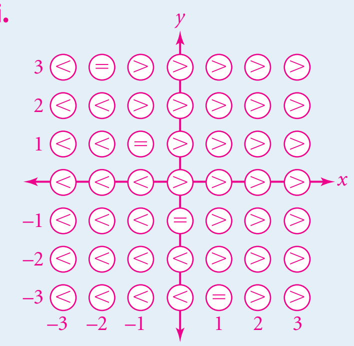
Step 1 If a group has only three members, have them reserve statement iv until they've finished the others. In a group with two members, each could take two statements, such as i and iii for one and ii and iv for the other.

Step 3 Graphs vary depending on statement chosen.

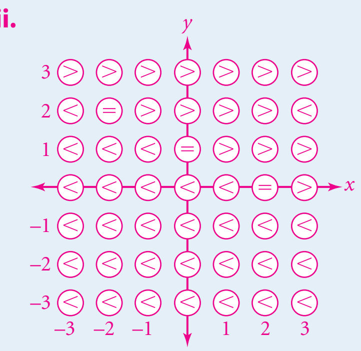
i.



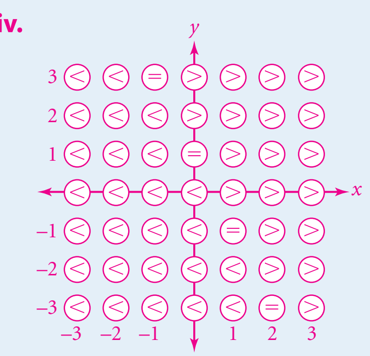
ii.



iii.



iv.



SHARING IDEAS

Ask students to describe the rules they derived for deciding which half-plane to shade, and prompt the class to critique them. Elicit the idea that the point (0, 0) is the easiest point to check and that doing so will give the desired information unless the line representing the equation passes through the origin.

Assessing Progress

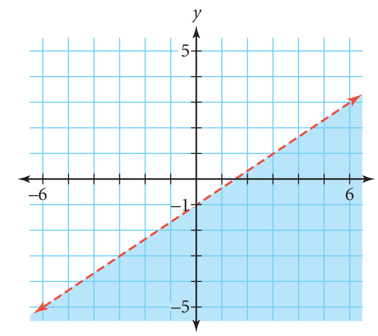
From your observations, you can assess students' understanding of inequality symbols and the number line, especially the fact that larger negative numbers are closer to 0. Also look for the ability to collect data systematically, to find patterns, and to work with a group.

EXAMPLE A

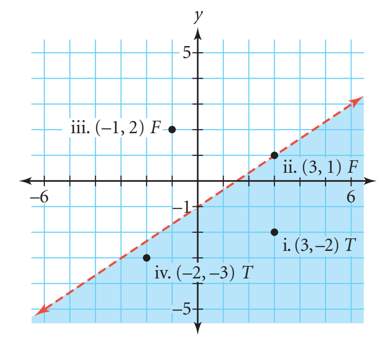
This example is good for students who had difficulty understanding the investigation. It also shows how to use calculators to evaluate the truth or falsity of statements.

You may need to remind students that points *satisfy* the inequality if they make the inequality true. They may check the potential solutions by hand as well as with a calculator.

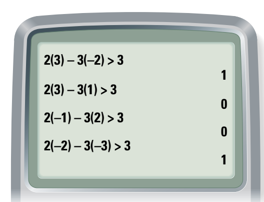
Graph the line $y = -1 + \frac{2}{3}x$ with a dashed line to indicate that points on the line are not part of the solution to the inequality. Because the inequality in y is less than the expression in x on the right side, shade the region *below* the line. Points in this region will have y-values that are less than the expression in x.



If you plot the given points, you'll see that the points that satisfy the inequality lie in the shaded part of the plane.



To check numerically whether the given points satisfy the inequality, substitute the x- and y-values from each given coordinate pair for x and y in the inequality $2x - 3y > 3$, and enter the inequality into your calculator. When you press ENTER, you'll see 1 if the inequality is true or 0 if the inequality is false, as shown on the calculator screen below. [▷ See **Calculator Note 5B.** ◁]



i. $2(3) - 3(-2) > 3 \longrightarrow 12 > 3 \longrightarrow$ True

ii. $2(3) - 3(1) > 3 \longrightarrow 3 > 3 \longrightarrow$ False

iii. $2(-1) - 3(2) > 3 \longrightarrow -8 > 3 \longrightarrow$ False

iv. $2(-2) - 3(-3) > 3 \longrightarrow 5 > 3 \longrightarrow$ True

Graphing Inequalities

- Draw a broken or dashed line on the boundary for inequalities with $>$ or $<$.
- Draw a solid line on the boundary for inequalities with $\geq$ or $\leq$.
- To graph inequalities in the form $y <$ or $y \leq$, shade below the boundary line.
- To graph inequalities in the form $y >$ or $y \geq$, shade above the boundary line.

[▷ See **Calculator Note 5C** to graph inequalities in two variables on your calculator. ◁]

EXAMPLE B

Graph and shade each inequality.

a. $x > -2$

b. $3y \leq 1$

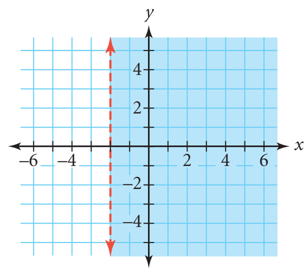
c. $-2x \geq 5$

d. $3 - y < 7$

▶ Solution

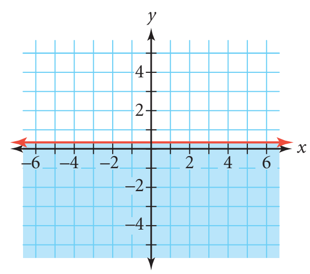
Solve for the variable in each inequality. Don't forget to switch the direction of the inequality when dividing by a negative!

a. $x > -2$



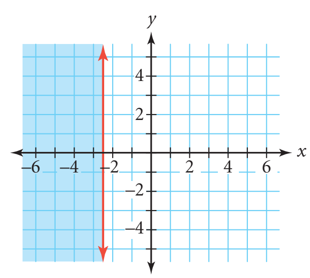
b. $3y \leq 1$ Divide each side by 3.

$y \leq \frac{1}{3}$



c. $-2x \geq 5$ Divide each side by -2.

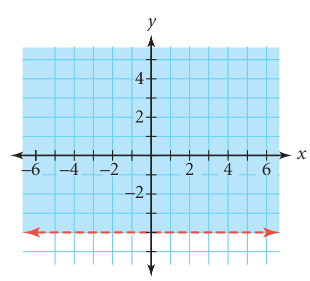
$x \leq -\frac{5}{2}$ or -2.5



d. $3 - y < 7$ Subtract 3 from each side.

$-y < 4$ Multiply each side by -1.

$y > -4$



EXAMPLE B

This example gives students more experience with two-dimensional graphs of inequalities in one variable. Special cases like this may be notationally simpler than inequalities involving two variables, but they are conceptually more difficult for many students.

Closing the Lesson

To visualize solutions to inequalities involving two variables, you can graph the line that is represented by the equation and then shade in the side of that line (**half-plane**) on which the solutions to the inequality lie.

BUILDING UNDERSTANDING

Students practice working with inequalities in two variables.

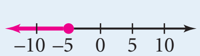
ASSIGNING HOMEWORK

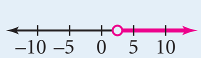
Essential	1–3, 4 or 5, 6–9
Performance assessment	6–8
Portfolio	8, 10
Journal	8, 13
Group	10, 11
Review	12–14

Helping with the Exercises

Exercise 1 If students are having difficulty, suggest that they turn the inequality into an equation and graph the line that is the boundary for the solutions to the inequality. Some students may want to solve for y to have the equation in intercept form.

Exercise 3 This exercise reviews graphing inequalities on a number line. You may want to ask students to discuss the difference between inequalities graphed on a number line and those graphed on a coordinate plane. **[Ask]** "How would you graph an inequality like $x < 5$ on coordinate axes?" [Shade the half-plane to the left of the vertical line $x = n$ if $x < n$ or to the right of the line if $x > n$.]

3a.


3b.


3c.


3d.


Exercises 4–6 **[Alert]** Check that students are using a broken line for strict inequalities and a solid line for others.

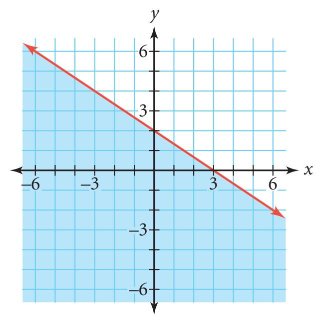
EXERCISES

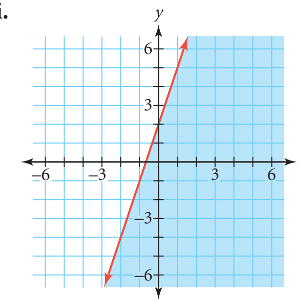
You will need your graphing calculator for Exercise **11.**

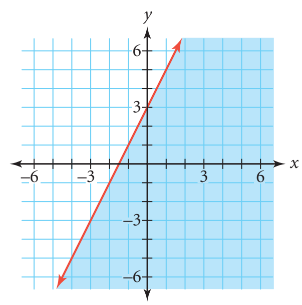
Practice Your Skills

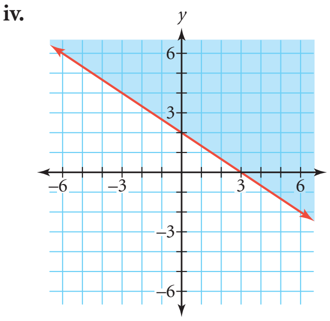
1. Match each graph with an inequality.

a. $y \le 3 + 2x$ iii **b.** $y \le 2 + 3x$ ii **c.** $2x + 3y \le 6$ @ i **d.** $2x + 3y \ge 6$ iv

i.


ii.


iii.


iv.


2. Solve each inequality for y.

a. $84x + 7y \ge 70$ @ $y \ge -12x + 10$ **b.** $4.8x - 0.12y < 7.2$ $y > 40x - 60$

3. Sketch each inequality on a number line.

a. $x \le -5$ **b.** $x > 2.5$ **c.** $-3 \le x \le 3$ @ **d.** $-1 \le x < 2$

4. Consider the inequality $y < 2 - 0.5x$.

a. Graph the boundary line for the inequality on axes scaled from -6 to 6 on each axis. @

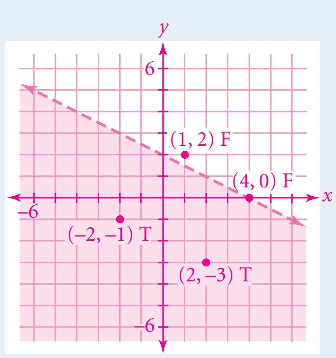
b. Determine whether each given point satisfies $y < 2 - 0.5x$. Plot the point on the graph you drew in 4a. Label the point T (true) if it is part of the solution or F (false) if it is not part of the solution region. @

i. (1, 2) **ii.** (4, 0) **iii.** (2, −3) **iv.** (−2, −1)

c. Use your results from 4b to shade the half-plane that represents the inequality. @

5. Consider the inequality $y \ge 1 + 2x$.

a. Graph the boundary line for the inequality on axes scaled from -6 to 6 on each axis.

4a–c.


b. Determine whether each given point satisfies $y \geq 1 + 2x$. Plot the point on the graph you drew in 5a, and label the point T (true) if it is part of the solution or F (false) if it is not part of the solution region.

i. $(-2, 2)$ **ii.** $(3, 2)$ **iii.** $(-1, -1)$ **iv.** $(-4, -3)$

c. Use your results from 5b to shade the half-plane that represents the inequality.

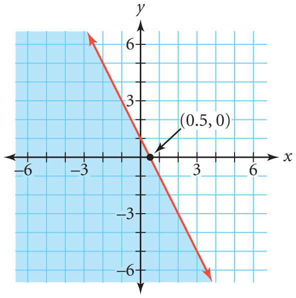
Reason and Apply

6. Sketch each inequality.

a. $y \leq -3 + x$ ⓐ **b.** $y > -2 - 1.5x$ **c.** $2x - y \geq 4$

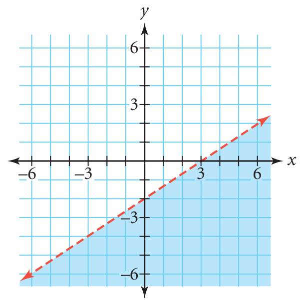
7. Write the inequality for each graph. ⓗ

a. ⓐ



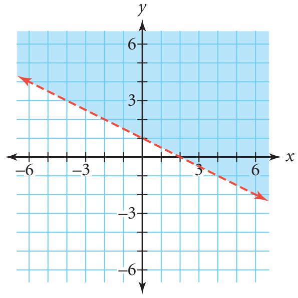
$y \leq 1 - 2x$

b.



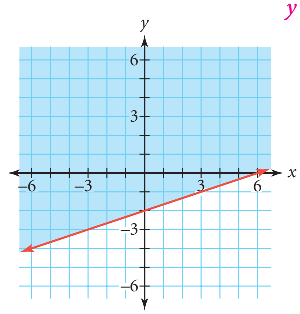
$y < -2 + \frac{2}{3}x$

c.



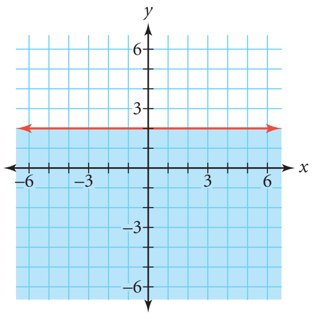
$y > 1 - 0.5x$

d.



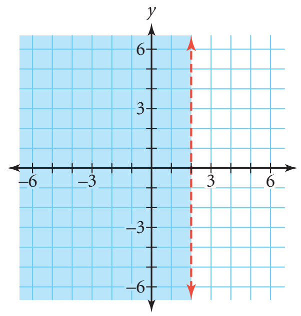
$y \geq -2 + \frac{1}{3}x$

e. ⓐ



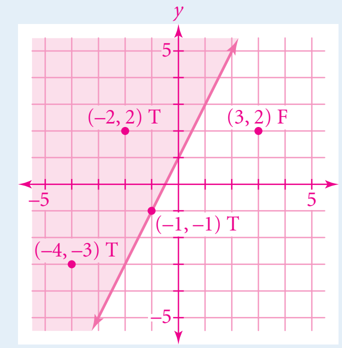
$y \leq 2$

f.

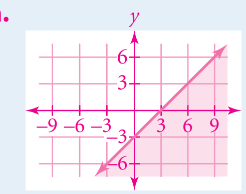


$x < 2$

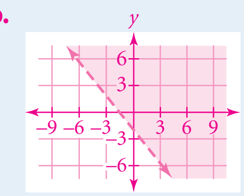
5a–c.



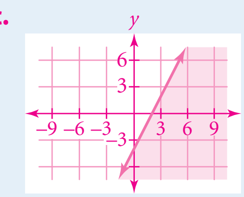
6a.



6b.

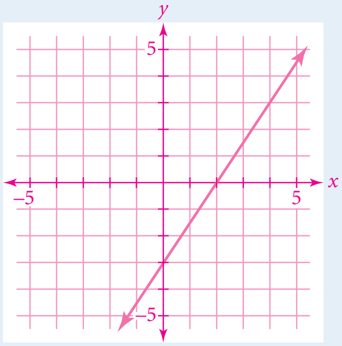


6c.

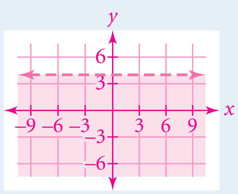


Exercise 8c Discourage students from memorizing a rule here. Checking a particular point, especially (0, 0) when the boundary line doesn't go through the origin, can be more meaningful.

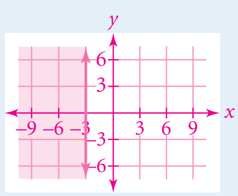
8a. $y = -3 + 1.5x$



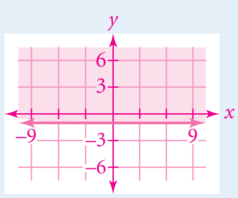
9a.



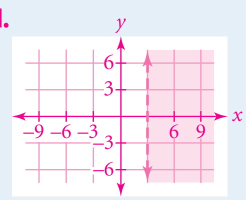
9b.



9c.



9d.



Exercise 10 Be sure students go through the entire modeling cycle for this real-world problem. In particular, they should consider how their solution to the inequality applies to the original situation. Though all points below the line are shaded to represent the solution set of the inequality, the only meaningful solutions to the problem are points representing nonnegative integer values for free throws and two-point shots.

8. ***Mini-Investigation*** Consider the inequality $3x - 2y \leq 6$.

a. Solve the equation $3x - 2y = 6$ for y and graph the equation.

b. Test the points $(1, 3)$ and $(1, -3)$. Which point makes the statement true? Does this indicate that you should shade above or below the line $3x - 2y = 6$? $(1, 3)$; above

c. You might think that the inequality $3x - 2y \leq 6$ indicates that you should shade below the boundary line. Make a conjecture about when you must shade the side that is opposite what the inequality symbol implies. If the coefficient of y is negative, then shade the side opposite what the inequality symbol indicates.

9. Sketch each inequality on coordinate axes.

a. $y < 4$ @ **b.** $x \leq -3$ **c.** $y \geq -1$ **d.** $x > 3$

10. **APPLICATION** The total number of points from a combination of one-point free throws, F, and two-point shots, S, is less than 84 points.

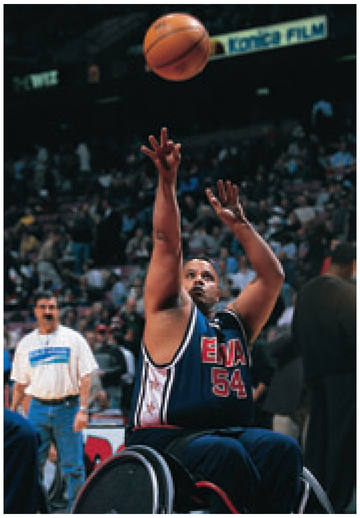
$F + 2S < 84$ **a.** Write an inequality to represent this situation. @

$F + 2S = 84$ **b.** Write the equation for the boundary line of this situation. @

c. Graph this inequality with S on the horizontal axis and F on the vertical axis. Show the scale on the axes.

d. On your graph, indicate three possible combinations of free throws and two-point shots that give a point total of 50. Label the coordinates of these points. possible answer: $(0, 50)$, $(10, 30)$, $(25, 0)$

11. Graph the inequalities in Exercises 4 and 5 on your calculator. [▶ See **Calculator Note 5C**. ◀]



Raul Acosta plays wheelchair basketball for the Eastern Paralyzed Veterans Association in New Jersey.

Review

4.2 **12.** These data are federal minimum wages of the past 70 years.

Federal Minimum Wages

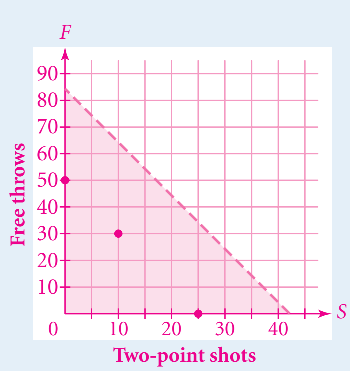
Year	Minimum wage	2003 equivalent dollars
1938	0.25	3.26
1939	0.30	3.97
1945	0.40	4.09
1950	0.75	5.73
1956	1.00	6.76
1961	1.25	7.69
1967	1.40	7.71
1968	1.60	8.46
1974	2.00	7.46
1975	2.10	7.18

Federal Minimum Wages

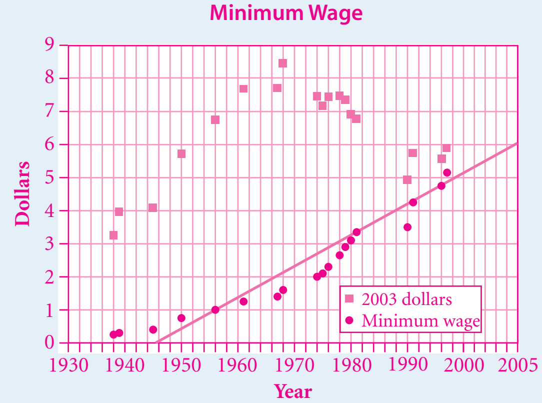
Year	Minimum wage	2003 equivalent dollars
1976	2.30	7.44
1978	2.65	7.48
1979	2.90	7.35
1980	3.10	6.92
1981	3.35	6.78
1990	3.50	4.93
1991	4.25	5.74
1996	4.75	5.57
1997	5.15	5.90

(Department of Labor, *www.dol.gov*)

10c.



12a, d.



a. Graph the data from the table on the same set of axes. Use one color for minimum wage and another for 2003 dollars.

b. Which is better represented by a line, the hourly minimum wage or the 2003 dollar value? minimum wage

c. Find the line of fit based on Q-points for the data points of the form (*year, minimum wage*). Q-points: (1956, 1.00), (1981, 3.35); $y = -182.864 + 0.094x$

d. Graph the equation in 12c to verify that it is a good fit.

e. What is the real-world meaning of the slope? How does it compare with the 2003 dollars graph?

2.5 **13.** Ellie was talking with her grandmother about a trip she took this summer. Ellie made the trip in 2.5 h traveling at 65 mi/h. Ellie's grandmother remembers that she made the same trip in about 6 h when she was Ellie's age. ⓗ

a. What speed was Ellie's grandmother traveling when she made the trip? about 27 mi/h

b. Explain how this is an application of inverse variation.

3.4 **14.** Solve each equation for y.

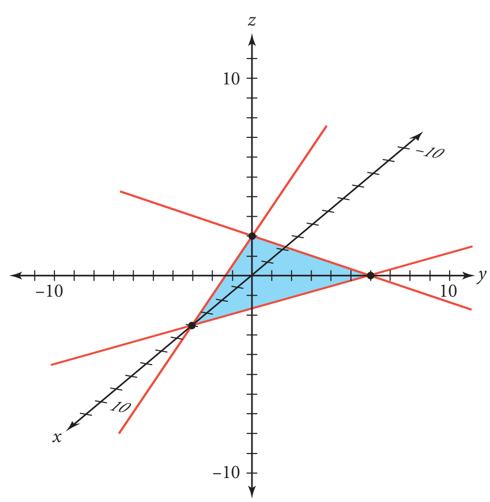
a. $7x - 3y = 22$ $\quad y = \frac{7}{3}x - \frac{22}{3}$

b. $5x + 4y = -12$ $\quad y = -\frac{5}{4}x - 3$

IMPROVING YOUR VISUAL THINKING SKILLS

In this chapter you have seen three possible outcomes for a system of two equations in two variables. If one solution exists, it is the point of intersection. If no solution exists, the lines are parallel and there is no point of intersection. If infinitely many solutions exist, the two lines overlap.

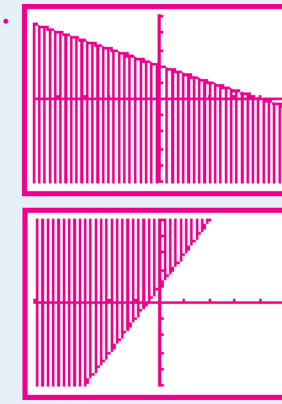
But what do the solutions look like in a system of three linear equations in three unknowns? An equation like $3x + 2y = 12$ is a line, but an equation in three variables is a plane. Consider the graph of $3x + 2y + 6z = 12$. Imagine the x-axis coming out of the page. The shaded triangle indicates the part of the solution plane whose coordinates are all positive. The complete plane is infinite.



If you have two more planar equations, you have a system of three equations in three variables. There will be three planes on the graph. So the solutions to this system are where the planes intersect, if they do at all. Visualize how three planes could intersect to answer these questions.

- Can three planes intersect in one point? If so, how many solutions will this system have?
- If a system has infinitely many solutions, must all three equations be the same plane?
- If the system has no solutions, must the planes be parallel?

11.



Exercise 12 This exercise is a follow-up to Chapter 4 Review, Exercise 10.

12e. The minimum wage increases 9¢ every year on average, but the actual dollar value was highest in 1968 and has decreased almost every year since then.

13b. Because $d = r \cdot t$ and the distance was the same for both Ellie and her grandmother, you can set these products equal to each other. If you let r represent Ellie's grandmother's speed, then $2.5(65) = 6r$.

IMPROVING VISUAL THINKING SKILLS

If two planes aren't parallel, they intersect in a line. If three planes all intersect, the intersection of each pair is a line. Those lines might intersect in a point, which means the system will have one solution. Or the three lines might be the same line, in which case the system has infinitely many solutions even though the planes aren't the same.

Or the lines of intersection of pairs of planes might all be parallel, extending the edges of a triangular prism. Then the system will have no solution, but the planes themselves won't be parallel. If two of the planes are parallel, there will also be no solution.

LESSON

5.7

PLANNING

LESSON OUTLINE

One day:

20 min	Investigation
15 min	Examples
5 min	Sharing
5 min	Closing
5 min	Exercises

MATERIALS

- Cereal Sales and Profit (T)
- Calculator Note 5C
- Sketchpad demonstration Linear Programming, *optional*

TEACHING

The solution set for a system of inequalities is the intersection of the solution sets for the individual inequalities.

One Step

Tell students that a cereal company is letting the buyer of each box of cereal enter a drawing for a $1,000 scholarship. One scholarship will be given away each month. The company makes a profit of between $0.47 and $1.10 on each box of cereal, depending on how the cereal is priced at different locations. If the company sells 2000 boxes in a month, will it make enough to cover the $1,000 scholarship? Ask students to answer the question in as many ways as they can. As you circulate, be sure that one way they answer is through a system of inequalities.

Guiding the Investigation

If you have a variety of envelopes that meet postal regulations, you might display them or even have

LESSON

5.7

Systems of Inequalities

All mathematical truths are relative, conditional.

CHARLES PROTEUS STEINMETZ

You learned that the solution to a system of two linear equations, if there is exactly one solution, is the coordinates of the point where the two lines intersect. In this lesson you'll learn about **systems of inequalities** and their solutions. Many real-world situations can be described by a system of inequalities. When solving these problems, you'll need to write inequalities, often called **constraints,** and graph them. You'll then find a region, rather than a single point, that represents all solutions.

Translucent sheets of blue, red, and yellow intersect to form overlapping regions of new colors—orange, green, and purple.

Investigation
A "Typical" Envelope

The U.S. Postal Service imposes several constraints on the acceptable sizes for an envelope. One constraint is that the ratio of length to width must be less than or equal to 2.5, and another is that this ratio must be greater than or equal to 1.3.

Step 1 Let l represent the length in inches and w represent the width in inches; $\frac{l}{w} \le 2.5, \frac{l}{w} \ge 1.3$.

Step 2 $\begin{cases} l \le 2.5w \\ l \ge 1.3w \end{cases}$; inequality is not reversed because w takes on only positive values.

Step 1 Define variables and write an inequality for each constraint.

Step 2 Solve each inequality for the variable representing length. Decide whether or not you have to reverse directions on the inequality symbols. Then write a system of inequalities to describe the Postal Service's constraints on envelope sizes.

Step 3 Decide on appropriate scales for each axis and label a set of axes. Decide if you should draw the boundaries of the system with solid or dashed lines. Graph each inequality on the same set of axes. Shade each half-plane with a different color or pattern.

Step 4 Where on the graph are the solutions to the system of inequalities? Discuss how to check that your answer is correct. Answers will vary. Check by substituting coordinates from the overlapping regions to see if they satisfy both inequalities.

LESSON OBJECTIVES

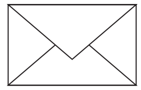
- Solve systems of inequalities by graphing
- Interpret the mathematical solutions in terms of the problem context
- Write inequalities to represent *constraints* in application problems

NCTM STANDARDS

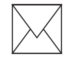
CONTENT		PROCESS	
	Number	✓	Problem Solving
✓	Algebra	✓	Reasoning
✓	Geometry	✓	Communication
	Measurement	✓	Connections
	Data/Probability	✓	Representation

Step 5 Decide if each envelope satisfies the constraints by locating the corresponding point on your graph.

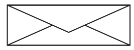
a. 5 in. by 8 in. yes



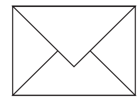
b. 3 in. by 3 in. no



c. 2.5 in. by 7.5 in. no



d. 5.5 in. by 7.5 in. yes



Step 6 Do the coordinates of the origin satisfy this system of inequalities? Explain the real-world meaning of this point. What constraints can you add to more realistically model the Postal Service's acceptable envelope sizes? How do these additions affect the graph?

EXAMPLE A

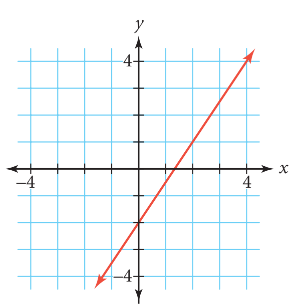
Graph the system of inequalities

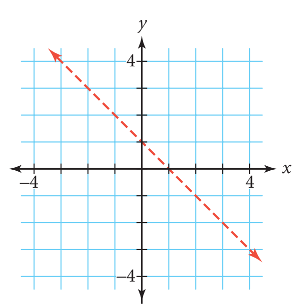
$$\begin{cases} y \leq -2 + \frac{3}{2}x \\ y > 1 - x \end{cases}$$

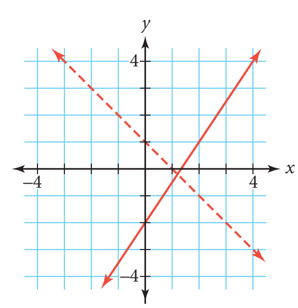
Graph the boundary lines and shade the half-planes. Indicate the solution area as the darkest region.

▶ *Solution*

First, determine if the boundary lines are solid or dashed. Graph $y = -2 + \frac{3}{2}x$ with a solid line because points on the line satisfy the inequality. Graph $y = 1 - x$ with a dotted line because its points do not satisfy the inequality.



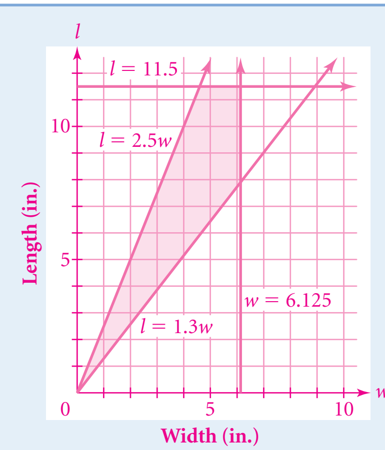




Shade the half-plane below the solid line $y = -2 + \frac{3}{2}x$ because its inequality has the "less than or equal to" symbol, $\leq$. Shade above the dotted line $y = 1 - x$ because its inequality has the "greater than" symbol, $>$. Use different colors or patterns to distinguish each area shaded.

Step 6 Answers will vary. Yes, (0, 0) satisfies the system. This means the envelope has no length or width. Minimum and maximum lengths and widths could be added as constraints. For example, the Postal Service lists $11\frac{1}{2}$ in. and $6\frac{1}{8}$ in. as the maximum length and width for an envelope with a 37¢ stamp. A sample system:

$$\begin{cases} l \leq 2.5w \\ l \geq 1.3w \\ w \leq 6.125 \\ l \leq 11.5 \end{cases}$$



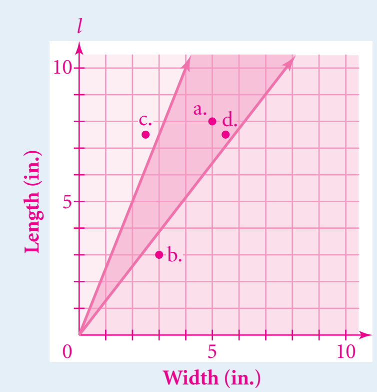
students measure them, report their ratios, and try to guess the legal constraints.

Step 2 You may need to remind students to multiply each side of the inequality by w. Encourage the habit of thinking about the sign of the multiplier of an inequality. Here, if nothing were known about w, you'd have to consider the case of negative w as well as positive w. Because w represents the width of an envelope, w is positive, so the direction of the inequality isn't changed.

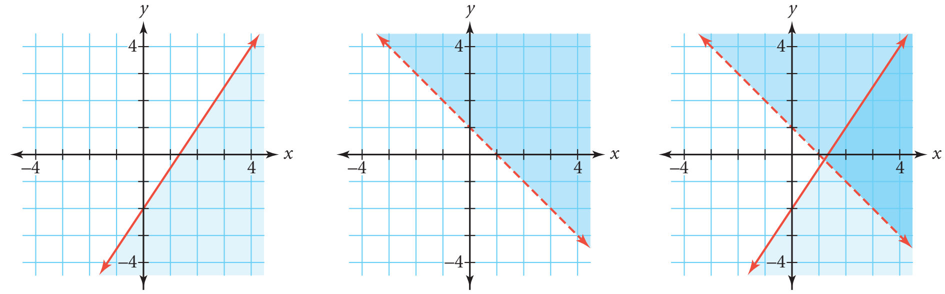
Step 3 It is often good for students to graph on a larger region than is needed. If they graph on a calculator, they may not be able to shade differently.

Step 3

Step 5



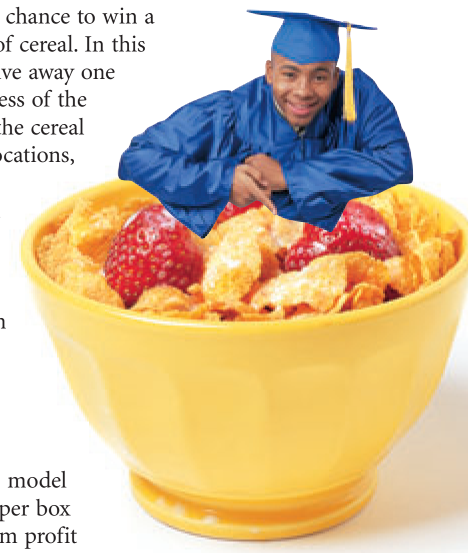
Step 6 For information about other constraints, visit www.keymath.com/DA and see the link to the U.S. Postal Service. Tell students that, although the units in this ratio cancel, the Postal Service uses "inches" as its unit of measurement.



Each shaded area indicates the region of points that satisfy each inequality. The overlapping area bounded by $y \le -2 + \frac{3}{2}x$ and $y > 1 - x$ satisfies both. Only the points that lie in both half-planes are the solutions to the system of inequalities.

EXAMPLE B

A cereal company is including a chance to win a \$1,000 scholarship in each box of cereal. In this promotional campaign, it will give away one scholarship each month, regardless of the number of boxes sold. Because the cereal is priced differently at various locations, the profit from a single box is between \$0.47 and \$1.10. Graph the expected profit, given the initial cost of the scholarship, for up to 5000 boxes sold in a month. Show the solution region on a graph. Is it possible to sell 3000 boxes and make a profit of \$1,000?



▶ Solution

Write a system of inequalities to model this situation. The lowest profit per box is \$0.47. So $0.47x$ is the minimum profit when x boxes are sold. Subtract \$1,000 for the scholarship given each month. So the profit y is at least $0.47x - 1000$ dollars for x boxes sold. This is given by the inequality

$$y \ge -1000 + 0.47x$$

Likewise, if the maximum profit is \$1.10 per box, then the profit is at most $1.1x - 1000$ dollars. So the second inequality is

$$y \le -1000 + 1.1x$$

The profit during each month is given by the system

$$\begin{cases} y \ge -1000 + 0.47x \\ y \le -1000 + 1.1x \end{cases}$$

SHARING IDEAS

Have students briefly share their answers to Step 5 and then discuss how well the solution to the system applies to the real-world problem. If you or your students have looked up more constraints, bring those out.

Assessing Progress
Students will show their ability at graphing the solution set for an inequality in two variables and their understanding of the need to reverse the direction of an inequality when multiplying by a negative number.

EXAMPLE A

This example shows how to solve an abstract system of inequalities.

[Ask] "In how many points can two half-planes intersect?" [infinitely many if they intersect, and zero if they don't]

Analogously, a system of linear *equations* has zero solutions, one solution, or infinitely many solutions because two lines intersect in zero points, one point, or infinitely many points.

Students enjoy watching their calculators graph a system of inequalities. If you have a projection panel, you might demonstrate.

Students may wonder if the intersection of two half-planes is a "quarter-plane." Because the area of the intersection is infinite, it isn't really smaller than a half-plane in any measurable sense.

EXAMPLE B

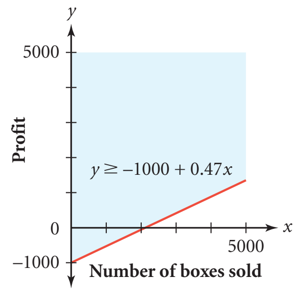
This example provides a real-world application of a system of inequalities. If students don't understand how to set up the inequalities, ask what they want to find (to help them determine the variables) and what they know (to write the inequalities).

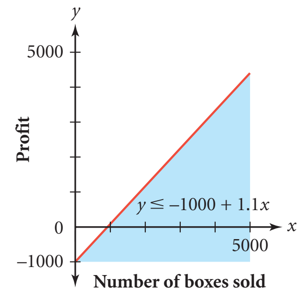
You might ask why anyone would want to know if it's possible to sell 3000 boxes and make \$1,000. For example, the marketing staff might want to know if the \$1,000 scholarship will be paid for by selling a projected number of boxes.

Pick various points in the solution region. Ask why they are solutions to each inequality and what real-world meaning they have. You might generate some good discussion if you pick a point with fractional coordinates. Students will need to realize that x is a number of boxes sold, so it must be an integer. The variable y is in dollars, so it can be fractional to an extent.

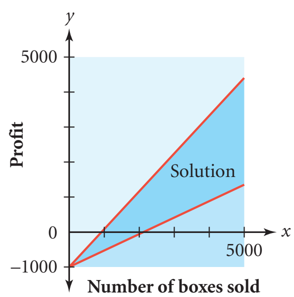
The Cereal Sales and Profit transparency shows the two graphs separately. If you cut the transparency apart, you can lay one graph on top of the other and get a result that looks like the third graph on page 323.

Each inequality is graphed for up to 5000 boxes on separate axes below.



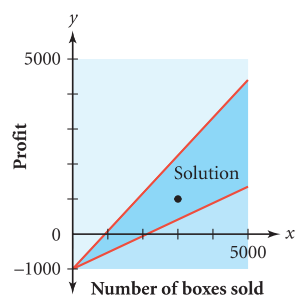


The possible profits are in the region where the two half-planes overlap. [▶ See **Calculator Note 5C** to graph systems of inequalities on your calculator. ◀]



To see if it is possible to make $1,000 when 3000 boxes are sold, plot the point (3000, 1000) on the graph. The point is in the solution region, so the coordinates satisfy both inequalities.

You can also substitute 3000 for x and 1000 for y and see if you get true statements.



$$\begin{array}{lcl} y \geq -1000 + 0.47x & & y \leq -1000 + 1.1x \\ 1000 \,\square\, -1000 + 0.47(3000) & \text{and} & 1000 \,\square\, -1000 + 1.1(3000) \\ 1000 \geq 410 & & 1000 \leq 2300 \end{array}$$

Both inequalities are true, so it is possible to sell 3000 boxes and make $1,000.

With enough constraints the solution to a system of inequalities might resemble a geometric shape or polygon. No matter how small the region, there are infinitely many points that satisfy the system. In some cases, the solution to a system of inequalities might be only a line or a line segment, but a line or segment still represents infinitely many solutions. It is also possible for the solution region to be merely a single point, or for there to be no solution region at all.

Closing the Lesson

You can find the solutions to a **system of inequalities** by finding the half-planes that give the solutions to the individual inequalities and then looking at their intersection.

BUILDING UNDERSTANDING

Students work with systems of inequalities. See Calculator Note 5D for using calculators on these exercises.

ASSIGNING HOMEWORK

Essential	1–5, 8, 10
Performance assessment	8, 12
Portfolio	8, 11
Journal	7
Group	7, 11, 12
Review	13–16

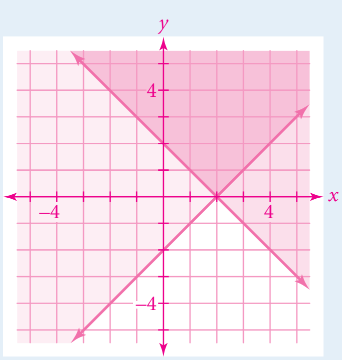
Helping with the Exercises

Exercise 2 Students might check whether pairs give solutions by substituting into the inequalities rather than graphing.

2a. Yes; (1, 2) satisfies both inequalities.

2b. No; only one inequality is satisfied.

3b.



EXERCISES

Practice Your Skills

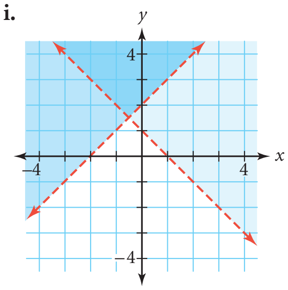
1. Match each system of inequalities with its graph.

a. $\begin{cases} y < 3 \\ x \geq 2 \end{cases}$ iii

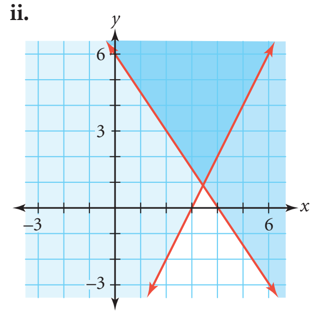
b. $\begin{cases} y > 2 + x \\ y > 1 - x \end{cases}$ i

c. $\begin{cases} 2x - y \leq 6 \\ 3x + 2y \geq 12 \end{cases}$ ii

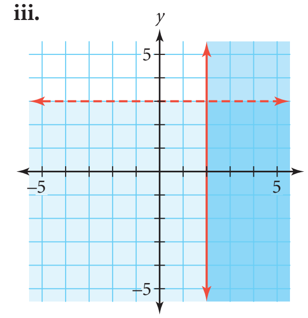
i.



ii.

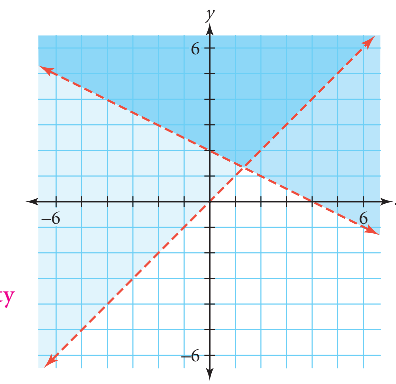


iii.



2. Here is the graph of this system of inequalities:

$$\begin{cases} y > x \\ y > 2 - \frac{1}{2}x \end{cases}$$



Is each point listed a solution to the system? Explain why or why not.

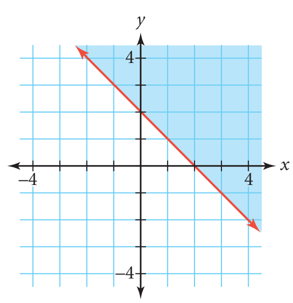
a. (1, 2) @

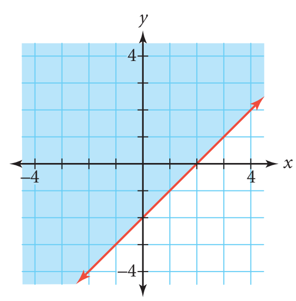
b. (3, 2)

c. $\left(\frac{4}{3}, \frac{4}{3}\right)$ No; for both inequalities, $\frac{4}{3} > \frac{4}{3}$ is not true.

d. (5, −3) No; neither inequality is satisfied.

3. Consider these two inequalities together as a system.





a. Name the inequality pictured in each graph. @ $y \geq -x + 2$; $y \geq x - 2$

b. Sketch a graph showing the solution to this system.

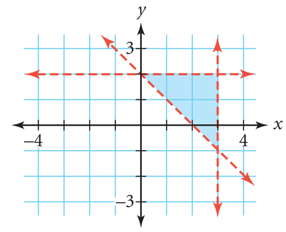
4. Sketch a graph showing the solution to each system.

a. $y \le 2$
$x < 2$

b. $x + y \le 4$
$x - y \le 4$

5. Write a system of inequalities for the solution shown on the graph. ⓗ

$\begin{cases} y > 2 - x \\ y < 2 \\ x < 3 \end{cases}$



Reasoning and Apply

6. **APPLICATION** The cereal company from Example B decides to raise the scholarship amount to $1,250. It also lowers the cereal's price so that the expected profit from a single box is between $0.40 and $1.00.

a. Write the inequalities to represent this new situation. ⓐ $y \ge -1250 + 0.40x, y \le -1250 + 1.00x, x \ge 0$

b. Graph the expected revenue for up to 5000 boxes sold in a month. ⓐ

7. **APPLICATION** On Kids' Night, every adult admitted into a restaurant must be escorted by at least one child. The restaurant has a maximum seating capacity of 75 people.

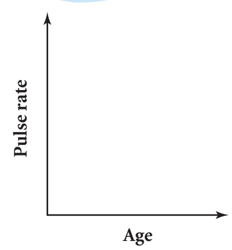
a. Write a system of inequalities to represent the constraints in this situation. ⓐ

7a. $\begin{cases} A \le C \\ A + C \le 75 \\ A \ge 0 \\ C \ge 0 \end{cases}$

b. Graph the solution. Is it possible for 50 children to escort 10 adults into the restaurant?

c. Why might the restaurant reconsider the rules for Kids' Night? Add a new constraint to address these concerns. Draw a graph of the new solution.

8. **APPLICATION** The American College of Sports Medicine considers age as one factor when it recommends low and high heart rates during workout sessions. For safe and efficient training, your heart rate should be between 55% and 90% of the maximum heart rate level. The maximum heart rate is calculated by subtracting a person's age from 220 beats per minute.



a. Define variables and write an equation relating age and maximum heart rate during workouts.

b. Write a system of inequalities to represent the recommended high and low heart rates during a workout. ⓐ

c. Graph the solution to show a region of safe and efficient heart rates for people of any age.

d. What constraints should you add to limit your region to show the safe and efficient heart rates for people between the ages of 14 and 40? ⓐ $a \ge 14$ and $a \le 40$

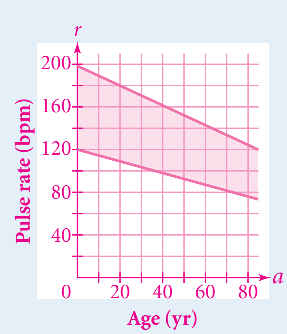
e. Graph the new solution for 8d.

7c. Answers will vary. It is possible to have all children and no adults at the restaurant. One possible additional constraint is that there must be at least one adult per five children, or $A \ge \frac{1}{5}C$. The solution for this set of constraints is the triangular region bounded by $A \le C$, $A + C \le 75$, and $A \ge 0.2C$.

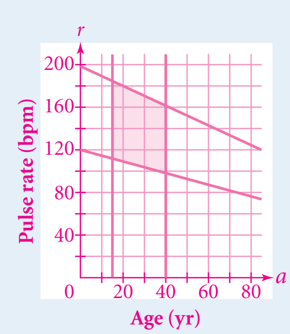
8a. $r = 220 - a$, where a represents age in years and r represents the heart rate in beats per minute

8b. $\begin{cases} r \le 0.90(220 - a) \\ r \ge 0.55(220 - a) \end{cases}$ or $\begin{cases} r \le 198 - 0.90a \\ r \ge 121 - 0.55a \end{cases}$

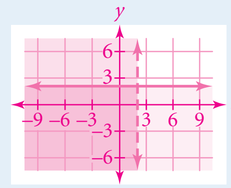
8c.



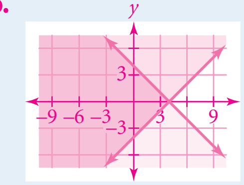
8e.



4a.

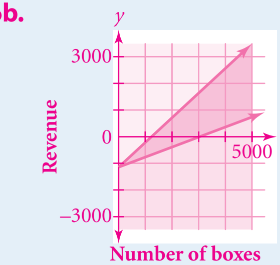


4b.



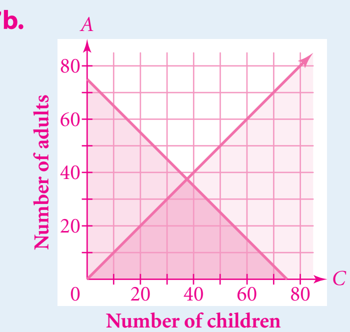
Exercise 6 Don't mark students wrong if they don't include the inequality $x \ge 0$. It is assumed but not stated in the example.

6b.



Exercise 7 A common error is to translate "every adult must be escorted by at least one child" as $A \ge 1C$ or $A \ge C$. Rather than just announcing that the inequality is backward, try to engage students in a conversation about how A and C represent numbers of adults and children, not just the words *adult* and *child*. Students may not list the inequalities $A \ge 0$ and $C \ge 0$. The Sketchpad demonstration Linear Programming can be used to replace this exercise.

7b.

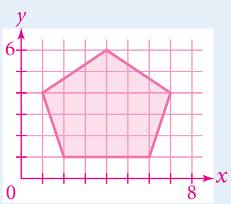


All the points in the dark-shaded triangular region satisfy the two inequalities. The point (50, 10) represents the situation in which 50 children escort 10 adults.

Exercises 9 and 10 Some students may find it difficult to work backward from the region to the inequalities, especially because they first have to find the equations of the lines. Suggest that they first write down a list of the steps and then follow their plan.

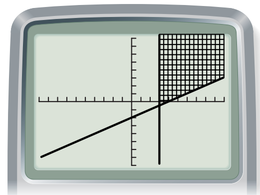
10. AB: $y \le \frac{2}{3}x + \frac{5}{3}$; BC: $y \le -\frac{3}{5}x + \frac{59}{5}$; AC: $y \ge \frac{1}{11}x + \frac{31}{11}$

11. The region is a pentagon.



keymath.com/DA

9. Write two inequalities that describe the shaded area below. Assume that the boundaries are solid lines and that each grid mark represents 1 unit.

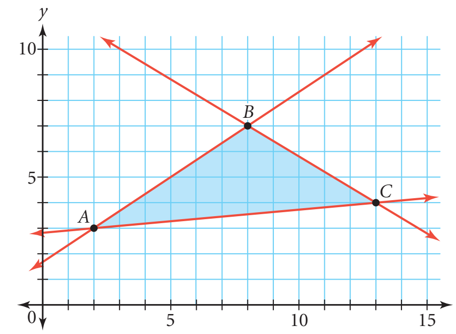


$x \ge 3$ and $y \ge -2 + \frac{1}{2}x$

10. Write a system of inequalities to describe the shaded area on the graph at right. Write each slope as a fraction. ⓗ

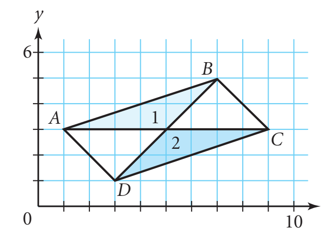
11. Graph this system of inequalities on the same set of axes. Describe the shape of the region.

$$\begin{cases} y \le 4 + \frac{2}{3}(x-1) \\ y \le 6 - \frac{2}{3}(x-4) \\ y \ge -17 + 3x \\ y \ge 1 \\ y \ge 7 - 3x \end{cases}$$



12. Write a system of inequalities that defines each shaded region of parallelogram $ABCD$ in the graph at right. ⓗ

Region 1: $\begin{cases} y \ge 3 \\ y \ge x - 2 \\ y \le \frac{1}{3}x + \frac{8}{3} \end{cases}$ Region 2: $\begin{cases} y \le 3 \\ y \le x - 2 \\ y \ge \frac{1}{3}x \end{cases}$

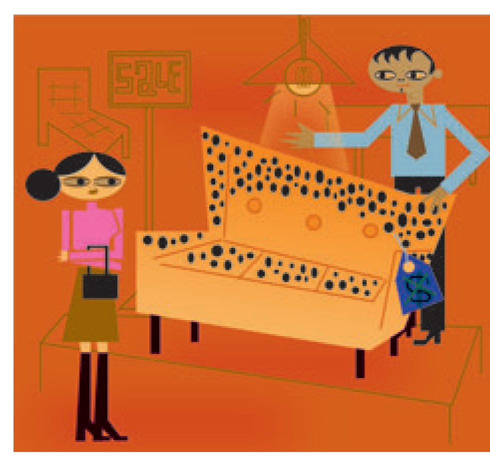


Review

2.3 **13.** **APPLICATION** Manuel has a sales job at a local furniture store. Once a year, on Employees' Day, every item in the store is 15% off regular price. In addition, salespeople get to take home 25% commission on the items they sell as a bonus.

a. A loft bed with a built-in desk and closet usually costs \$839. What will it cost on Employees' Day? ⓐ \$713.15

b. At the end of the day, Manuel's bonus is \$239.45. How many dollars worth of merchandise did he sell? ⓗ \$957.80



2.8 **14.** Think about the number trick shown at right.

a. Layla got a final number of 4. What was her original number? 4

b. Robert got a final answer of 10. What was his original number? 10

c. Let x represent the starting number. Write an algebraic expression to represent this sequence of operations. Then simplify the expression as much as possible.

x	____
Ans · 3	____
Ans + 12	____
Ans / 5	____
Ans − 1.4	____
Ans · 10	____
Ans − 10	____
Ans / 6	____

14c. $\dfrac{10\left(\frac{3x+12}{5} - 1.4\right) - 10}{6}$, which simplifies to x

5.3 **15.** Solve each system of equations by using a symbolic method. Check that your solutions are correct.

a. $\begin{cases} y = 4x - 3 \\ y = 2x + 9 \end{cases}$ $x = 6, y = 21$

b. $\begin{cases} 3x - 4y = -2 \\ -2x + 3y = 1 \end{cases}$ $x = -2, y = -1$

5.2 **16.** Mr. Diaz makes an organic weed killer by mixing 8 ounces of distilled white vinegar with 20 ounces of special-strength pickling vinegar. Distilled white vinegar is 5% acid and Mr. Diaz's mixture is 15% acid. What is the acid concentration of the pickling vinegar? 19% acid

IMPROVING YOUR REASONING SKILLS

Suppose 9 crows each make 9 caws 9 times throughout the day. How many total caws are there?

Suppose 99 crows make 99 caws 99 separate times in one day. Now how many caws are there?

Answer the question again for 999 crows making 999 caws 999 times. If you continue this pattern of problems, at what number does your calculator round the answer? What is the exact number of caws in this case?

Write the answers to the first three questions and look for a pattern. Use it to find how many caws there are when the number is 99,999. With 86,400 seconds in a day, this means that each crow makes more than one caw per second every hour!

IMPROVING REASONING SKILLS

The calculators referred to in the calculator notes will use scientific notation for numerals with more than ten digits. Some more powerful calculators, such as the TI-89, will hold many more digits. At 9,999 crows, the TI-83 begins rounding.

Crows	Caws
9	729
99	970,299
999	997,002,999
9,999	999,700,029,999
99,999	999,970,000,299,999

CHAPTER REVIEW

5

PLANNING

LESSON OUTLINE

One day:

- **5 min** Introduction
- **15 min** Exercises and helping individuals
- **15 min** Checking work and helping individuals
- **15 min** Student self-assessment

REVIEWING

Use Exercise 13 from Lesson 5.3 to review the methods for solving systems. Ask students to solve each equation for one variable and graph the system and then to solve by substitution. Have students then solve the system by elimination, and if you are covering matrices, have students solve by matrix row operations as well. Then change the problem to say that Crystal and Dan must buy at least 12 pictures total to give to friends and family, but they only have \$84 to spend. Have students write and graph a system of inequalities to represent this problem. Then have them find the possible combinations of wallet- and portrait-size pictures they can purchase. **[Ask]** "If Crystal and Dan purchase only wallet-size pictures, how many can they buy? How about if they each buy one portrait-size picture?" [They can buy from 12 to 25 wallet-size pictures if they don't buy any portrait-size pictures. If they purchase a total of two portrait-size pictures (one each), they can buy from 10 to 19 wallet-size pictures.]

CHAPTER 5 REVIEW

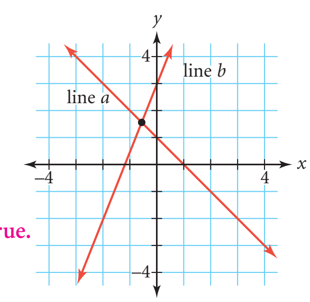
In this chapter you learned to model many situations with a **system of equations** in two variables. You learned that systems of linear equations can have zero, one, or infinitely many solutions. You used tables, used graphs, and solved symbolically to find the solutions to systems. You discovered that the methods of **elimination, substitution,** and **row operations** on a matrix allow you to find exact solutions to problems, not just the approximations of graphs and tables.

Then you analyzed situations involving **inequalities** and discovered how to find their solutions using graphs, tables, and symbolic manipulation. The graph of an inequality in one variable is a part of a number line, and the graph of a linear inequality in two variables is a shaded **half-plane** that contains points whose coordinates make the inequality true. A **compound inequality** is the combination of two inequalities.

You discovered how to use inequalities to define **constraints** that limit the solution possibilities in real-world applications. You learned how to graph a **system of inequalities.**

EXERCISES

@ Answers are provided for all exercises in this set.



1. Lines a and b at right form a system of equations. Write the equations of the lines and find the exact point of intersection.

2. Find the point where the graphs of the equations intersect. Check your answer.

$$\begin{cases} 3x - 2y = 10 \\ x + 2y = 6 \end{cases}$$

The lines meet at the point (4, 1); the equations $3(4) - 2(1) = 10$ and $(4) + 2(1) = 6$ are both true.

3. Graph this system of equations, and find the solution point.

$$\begin{cases} y = 5 - 0.5(x - 3) \\ y = -4 + 1.5(x + 2) \end{cases}$$

4. Show the steps involved in solving this system symbolically by the substitution method. Justify each step.

$$\begin{cases} y = 16 + 4.3(x - 5) \\ y = -7 + 4.2x \end{cases}$$

5. Complete each sentence.

a. A system of two linear equations has no solution if . . .
. . . the slopes are the same but the intercepts are different (the lines are parallel).

b. A system of two linear equations has infinitely many solutions if . . .
. . . the slopes are the same and the intercepts are the same (the lines coincide).

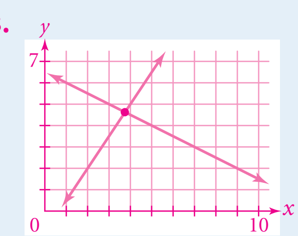
c. A system of two linear equations has exactly one solution if . . .
. . . the slopes are different (the lines intersect in a single point).

ASSIGNING HOMEWORK

Exercises 1–5 review the mechanics of solving systems of equations. Exercises 6–8 review inequalities and systems of inequalities. Exercise 9 is an application that you might want students to complete in groups. Assign Exercise 10 only if you are covering matrices.

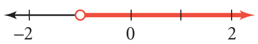
1. line a: $y = 1 - x$; line b: $y = 3 + \frac{5}{2}x$; intersection: $\left(-\frac{4}{7}, \frac{11}{7}\right)$

3.



The point of intersection is (3.75, 4.625).

6. Name the inequality that each graph represents.

a. 

$x > -1$

b. 

$x < 2$

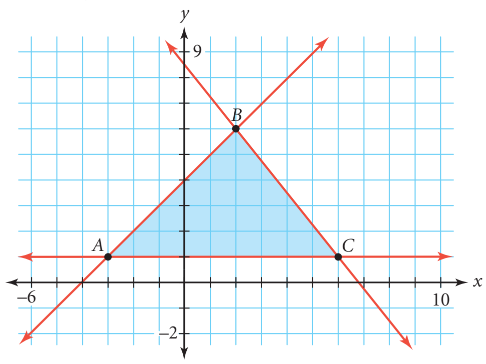
c. 

$-2 \le x < 1$

7. Solve the inequality $5 \le 2 - 3x$ for x and graph the solution on a number line.

$x \le -1$ 

8. Write a system of inequalities to describe this shaded area.



$$\begin{cases} y \le x + 4 \\ y \le -1.25x + 8.5 \\ y \ge 1 \end{cases}$$

9. **APPLICATION** Harold cuts lawns after school. He has a problem on Wednesdays when he cuts Mr. Fleming's lawn. His lawn mower has two speeds—at the higher speed he can get the job done quickly, but he always runs out of gas; at the lower speed he has plenty of gas, but it seems to take forever to get the job done. So he has collected this information.

- On Monday he cut a 15-meter-by-12-meter lawn at the higher speed in 18 minutes. He used a half tank of gas, or 0.6 liter.
- On Tuesday he cut a 20-meter-by-14-meter lawn at the lower speed in 40 minutes. He used a half tank of gas.
- Mr. Fleming's lawn measures 22 meters by 18 meters.

a. How many square meters of lawn can Harold cut per minute at the higher speed? At the lower speed? $10\ \text{m}^2/\text{min}$; $7\ \text{m}^2/\text{min}$

b. If Harold decides to cut Mr. Fleming's lawn using the higher speed for 10 minutes and the lower speed for 8 minutes, will he finish the job? No; he will cut $156\ \text{m}^2$, and the lawn measures $396\ \text{m}^2$.

c. Let h represent the number of minutes cutting at higher speed, and let l represent the number of minutes cutting at lower speed. Write an equation that models completion of Mr. Fleming's lawn. $10h + 7l = 396$

d. How much gas does the lawn mower use in liters per minute at the higher speed? At the lower speed? $\frac{1}{30}$ L/min; $\frac{3}{200}$ L/min

e. Write an equation in terms of h and l that has Harold use all of his gas. $\frac{h}{30} + \frac{3l}{200} = 1.2$

f. Using the equations from 9c and e, solve the system and give a real-world meaning of the solution. $l = 14.4$ min, $h = 29.52$ min; if Harold cuts for 29.52 min at the higher speed and 14.4 min at the lower speed, he will finish Mr. Fleming's lawn and use one full tank of gas.

10. Use row operations to find the solution matrix for this system.

$$\begin{cases} 7x + 3y = -45 \\ x + 6y = -51 \end{cases} \quad \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -8 \end{bmatrix}$$

4.

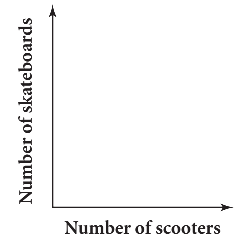
$16 + 4.3(x - 5) = -7 + 4.2x$	Set the right sides of the two equations equal to each other.
$16 + 4.3x - 21.5 = -7 + 4.2x$	Apply the distributive property.
$-5.5 + 4.3x = -7 + 4.2x$	Subtract.
$0.1x = -1.5$	Add $-4.2x$ and 5.5 to both sides.
$x = -15$	Divide both sides by 0.1.
$y = -7 + 4.2(-15)$	Substitute -15 for x to find y.
$y = -70$	Multiply and add.

The solution is $x = -15$ and $y = -70$.

TAKE ANOTHER LOOK

Businesses use systems of equations and inequalities to determine how to maximize profits. A process called **linear programming** applies the concepts of constraints, points of intersection, and algebraic expressions to solve this very real application problem. Here is one example.

A company manufactures scooters and skateboards. The factory has the capacity to make at most 6000 scooters in one day, and the factory can make at most 8000 skateboards in one day. However, the factory can produce a combination of no more than 10,000 scooters and skateboards together. Define variables and write a system of three inequalities to describe these constraints. Label a set of axes and graph the solution. This is called a **feasible region.** What do the points in this shaded region represent? Find the points of intersection at the corners of this region.



The company makes a profit of \$15 per scooter and \$10 per skateboard. How many of each should the company make to maximize its profits? To answer this question, use the variables defined earlier to write an expression to find the total profit the company makes from scooters and skateboards. Then substitute the coordinates of several points from the feasible region including the points of intersection. For example, if the company makes 5000 scooters and 5000 skateboards, substitute 5000 for x and 5000 for y into your expression to find the profit. Which point gave you the greatest profit?

Professional skateboarder Tony Hawk performs at the X Games.

Take Another Look

Linear programming problems like this one deal with many more variables and inequalities. An application might require that you maximize or minimize a linear function of the two variables, using only points in this region. The largest and smallest values of the function occur along the boundary of the region, usually at a corner. These problems are a major application of mathematics today.

$$\text{System: } \begin{cases} x \leq 6000 \\ y \leq 8000 \\ x + y \leq 10{,}000 \\ x \geq 0 \\ y \geq 0 \end{cases}$$

The profit is given by $15x + 10y$. At (6000, 4000), the profit is the maximum, \$130,000.

Assessing What You've Learned

In each of the five chapters from Chapter 0 to Chapter 4, you were introduced to a different way to assess what you learned. Maybe you have tried all five ways—keeping a portfolio, writing in your journal, organizing your notebook, giving a presentation, and doing a performance assessment. Maybe you have tried just a couple of these methods. Probably, your teacher has adapted these ideas to suit the needs of your class.

By now, you should realize that assessment is not just giving and taking tests. In the working world, performance in some occupations can be measured in tests, but in all occupations, there is a need to communicate what you know to coworkers. In all jobs, workers demonstrate to their employer or to their clients, patients, or customers that they are skilled in their fields. They need to show they are creative and flexible enough to apply what they've learned in new situations. Assessing your own understanding and letting others assess what you know gives you practice in this important life skill. It also helps you develop good study habits, and that, in turn, will help you advance in school and give you the best possible opportunities in your work life. Keep that in mind as you try one or more of these suggestions.

UPDATE YOUR PORTFOLIO Choose your best graph of a system of inequalities from this chapter to add to your portfolio. Redraw the graph with a clearly labeled set of axes. Use color to highlight each inequality and its half-plane. Indicate the solution region with a visually pleasing design or pattern.

WRITE IN YOUR JOURNAL Add to your journal by answering one of these prompts:

- You have learned five methods to find a solution to a system of equations. Which method do you like best? Which one is the most challenging to you? What are the advantages and disadvantages of each method?
- Describe in writing the difference between an inequality in one variable and an inequality in two variables. How do the graphs of the solutions differ? Compare these to the graph of a system of inequalities.

ORGANIZE YOUR NOTEBOOK Update your notebook with an example, investigation, or exercise that demonstrates each solution method for a system of equations. Add one problem that demonstrates each of these concepts: inequalities in one variable, inequalities in two variables, and systems of inequalities.

GIVE A PRESENTATION Write your own word problem for a system of equations or inequalities. Choose a setting that is meaningful to you or that you wish to know more about, and write a problem to model the situation. It can be about winning times for Olympic events, the point where two objects meet while traveling, percent mixture problems, or something new you created. Solve the problem using one of the methods you learned in this chapter. Make a poster of the problem and its solution, and present it to the class. Work with a partner or in a group.

PERFORMANCE ASSESSMENT As a classmate, family member, or teacher watches, solve a system of equations using at least two different methods. Explain your process, and show how to check your solution.

ASSESSING

For written assessment, use Constructive Assessment items for this chapter or one of the chapter tests from Assessment Resources. Or create your own test using the Test Generator CD, omitting matrices if you are not covering them.

FACILITATING SELF-ASSESSMENT

To help students complete the portfolio described in Assessing What You've Learned, suggest that they consider for evaluation their work in Lesson 5.1, Exercises 8 and 10; Lesson 5.2, Exercise 12; Lesson 5.3, Exercises 9 and 14; Lesson 5.4, Exercises 9–11; Lesson 5.5, Exercise 8; Lesson 5.6, Exercises 8 and 10; and Lesson 5.7, Exercises 8 and 11.